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ENGINEERING MATHEMATICS

WITH TABLES

BY

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WITH 260 ILLUSTRATIONS

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PREFACE

This book is the outgrowth of fifteen years of experience in the teaching of Mathematics to Evening School Engineering students—men who need sound training in Mathematics, but who must get this training in the shortest time consistent with a thorough understanding of the subject.

For this earnest type of student, whether in day or evening school, the so-called practical mathematics textbook is not satisfactory even though it be filled with the "practical problems," common to such books. The student preparing for Engineering needs a more thorough knowledge of Mathematics than such a book offers. On the other hand because he has no time to lose, it is not reasonable to expect him to spend any time in the study of mathematical principles or processes not directly essential to his further education in Engineering. For this reason the more lengthy and involved textbooks used in many colleges are not applicable.

What *is* needed is a book that covers the middle ground between the practical mathematical treatise on the one hand and the more elaborate college text on the other. Seeing the growing need for such a book, the Drexel Evening School executives have taken a very broad and helpful attitude in the preparation of this treatise. As a result, it has been worked out along rather unique, as well as logical, lines.

First, the book was completely written by the three authors with the assistance of the heads of the various Engineering Departments of the Evening School. These men, each with from ten to twenty instructors under him, furnished many of the engineering problems, and to a large extent determined the policy of the book by making sure that its contents were such as to prepare students properly for their advanced engineering classes.

Second, the book was put into mimeograph form and used by more than twenty instructors in Mathematics, experienced in the teaching of men whose time was valuable as well as limited. While teaching from the book, these instructors held frequent meetings with the authors, giving valuable suggestions and reports on the progress of the students using the book, as compared with the progress of similar students using former textbooks. At the end of this probation period the authors completely revised the book for publication.

This thorough organization of the book thus represents the composite efforts of over thirty men actively interested in its preparation. Each of these men had just one point in view—the preparation of the busy student for a thorough Engineering Course. Without the co-operation of the Drexel Evening School executives such a comprehensive plan could not have been carried out

The earlier chapters of the book form a natural transition from Arithmetic to Secondary Mathematics. The authors have found that such a transition gives the student a much clearer conception of algebraic processes, resulting in more rapid progress than is possible through the usual method of treating Algebra as an entirely separate subject.

In the succeeding chapters the material is so arranged as to correlate the algebraic, geometric, and trigonometric ideas which are too often treated separately. For example, throughout the book, algebraic processes have been developed to the point where they may be used naturally in solving geometric and trigonometric problems. This makes it easier for the student to grasp the various geometric and trigonometric relations, and not only facilitates considerably his understanding, but is an **incentive** to the study of Algebra itself. Classroom practice has proved that this added incentive and interest on the part of the student is much more vital to progress than is generally recognized.

The authors have not lost sight of the fact that the student preparing for Engineering must take more advance Mathematics than is covered by this book. With this in mind, they have selected problems of such a type as to furnish a logical transition from Secondary to Higher Mathematics.

To summarize—this book covers all the Mathematics essential to engineering preparation between Arithmetic and Analytical Geometry. Numerous problems have been included, because it is a commonly accepted fact that the problem solving method is the best known method of teaching Mathematics. Many of the more practical problems may be omitted at the discretion of the teacher without affecting the continuity of development of the subject.

The authors wish to express their appreciation of the invaluable suggestions and the enthusiasm of Mr. W. T. Spivey, Director of the Drexel Evening School and consulting engineer. They appreciate also the co-operation extended them by the Drexel Evening School management and the Heads of the Engineering Departments.

The authors are further indebted to their associate instructors in the Mathematics Department for valuable criticism, and to the skill of Mr. Wm. R. Miller, instructor in Mathematics, who made the drawings for all cuts used in the book.

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SYMBOLS

$+$	plus, increased by.	$\perp$	perpendicular, is perpendicular to.
$-$	minus, diminished by.	$\perp$s	perpendiculars.
$\times$	multiplied by.	$\parallel$	parallel, is parallel to.
$\div$	divided by.	$\parallel$s	parallels.
$=$	equals, is equal to.	$\triangle$	triangle.
$\equiv$	is identically equal to.	$\triangle$s	triangles.
$\neq$	is not equal to.	▱	parallelogram.
$\overset{m}{=}$	is measured by.	▱s	parallelograms.
$>$	is greater than.	○	circle.
$<$	is less than.	○s	circles.
§	article.	$\overset{\frown}{AB}$	arc AB.
$\propto$	varies as.	$\overline{AB}$	chord AB.
$\therefore$	therefore, hence.	$\Bumpeq$	is equivalent to.
. . .	and so on.	∞	infinity.
$\underline{\lvert n}$ or $n!$	$1 \cdot 2 \cdot 3 \cdots n$ (factorial n).	π	pi.
$\angle$	angle.	$^\circ \; ' \; ''$	degrees, minutes, seconds.
$\angle$s	angles.	$' \; ''$	feet, inches.

ABBREVIATIONS

Adj.	adjacent.	Hyp.	hypothesis.
Alt.	alternate.	*i.e.*	that is.
Art.	article.	Int.	interior.
Ax.	axiom.	Isos.	isosceles.
Comp.	complement.	Opp.	opposite.
Const.	construction.	Prob.	problem.
Def.	definition.	Pt.	point.
Ext.	exterior.	Rt.	right.
Fig.	figure.	St.	straight.
Geom.	geometric.	Sup.	supplement.
Q.E.D.	quod erat demonstrandum "which was to be proved."		
Q.E.F.	quod erat faciendum "which was to be made."		

ENGINEERING MATHEMATICS

CHAPTER I

GRAPHIC REPRESENTATION

1. **Related Values Shown Graphically.**—Any set of related numerical facts may be conveniently shown on a diagram. Such a diagram appeals to the eye more readily than a set of tabulated figures.

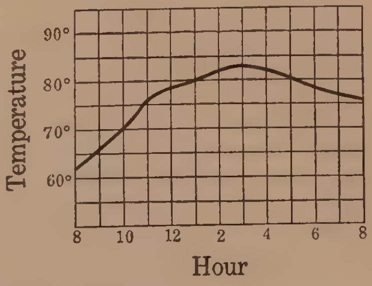
Each pair of related values locates one point. When a sufficient number of points has been located, a line drawn through the series of points shows at a glance the corresponding changes in the two related values compared. This diagram is commonly called a graph.

The graph is usually drawn on paper ruled in squares, known as **coordinate paper.** Two principal lines at right angles to each other are chosen as **axes** of reference. The point of intersection of the axes is called the **origin.** From the origin each point is located by measuring each pair of values along the horizontal and vertical axes respectively.

Illustrations. 1.—The record of the temperature in Philadelphia for April 9th, 1922, is as follows:

Time, A. M.	8	9	10	11	Noon	1	2	3	4	5	6	7	8
Temperature	62	66	70	76	78	80	82	83	82	80	78	77	76

To every hour there corresponds a temperature. These together give a pair of values for which we locate a point. (For every pair of values there will be one point.)



Let the origin represent 8 A. M., and 50° of temperature. Let each division on the horizontal axis represent one hour, and each division on the vertical axis represent 5°. Thus 10 A. M. is located two divisions to the right along the horizontal axis, and 70° is located four divisions up along the vertical axis. The point representing this pair of values is located by the intersection of the vertical line through 10 A. M. and the horizontal line through 70°.

Smooth Curve.—When a point is located for each pair of values, a smooth curve is drawn through all the points. A smooth curve is used because the change in temperature is gradual from hour to hour.

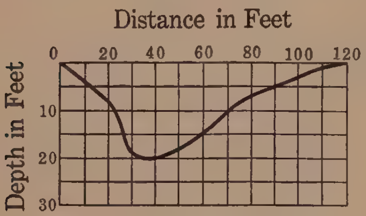
From the graph answer the following: Estimate the temperature at 9:30; at 10:15; at 12:45; at 5:30. When was the temperature highest?

When lowest? Between what hours was the greatest rise? The greatest fall? The least rise? The least fall?

2. Let us consider a stream 120 feet wide, across which a dam is to be built. The engineer must compute the amount of material and labor required before he can submit an estimate of the cost. To do this it will be necessary for him to know the depth of the stream at different intervals. He measures the depth of the water at intervals of 10 feet from the left bank. The depths are found to be as follows:

0, 4, 8, 19, 20, 18, 15, 10, 7, 5, 3, 1, 0 feet.

These figures mean much more to the engineer after he has drawn a graph representing a section of the bed of the stream. This graph is known as a profile and is plotted as follows: Using coordinate paper we take a horizontal line as representing the surface of the water and measure depths vertically below this line at intervals of 10′. Each division represents 10 feet for width and 5 feet for depth. To every horizontal distance there corresponds a depth. These together give a pair of values for which we locate a point. Next a line is drawn connecting all of the above points. This line is the profile of the bed of the stream. What is the greatest depth? Estimate the depth 15 feet from the left bank, 25′, 35′, 45′, 55′, 75′, 105′.



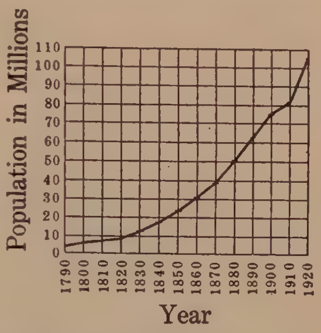
Between what points does the depth change most rapidly? Where least?

3. Again consider the mass of figures giving the population of the U. S. from 1790 to 1920 inclusive to the nearest hundred.

Year	Population	Year	Population
1790	3,929,200	1860	31,443,300
1800	5,308,500	1870	38,558,400
1810	7,239,900	1880	50,155,800
1820	9,638,500	1890	62,947,800
1830	12,860,700	1900	75,994,600
1840	17,063,400	1910	91,972,300
1850	23,191,900	1920	105,710,600

Represent the above statistics by a graph. On the horizontal axis let each division represent 5 years and on the vertical axis let each division represent 5 millions of population.

By looking at the resulting graph we see at a glance the change in the population for the entire period covered and visualize the change far better than by studying the tabulated values. The real meaning and value of such data as is tabulated in the above illustration is lost sight of in the mass of figures, whereas the graphical representation, although not giving the exact numerical values, enables one to readily visualize the whole range of relations.



2. Increasing Use of Graphs.—Whenever two related quantities are to be compared through a series of values, the graph may be used. Thus the broker may trace the rise and fall of stocks. The manufacturer can compare his gross earnings, his expenses, and net earnings on one sheet by using three different lines. The Weather Bureau shows variations of temperature, wind, etc., on a graph. The nurse indicates the variation of the patient's temperature on a chart, and the physician can follow it at a glance. The engineer uses the graph constantly. Text-books, newspapers, magazines, and trade journals use it freely. A little thought will convince the student that many of his daily activities depend on each other and that these relations may be shown graphically.

3. The Broken-line Graph.—The broken-line graph is made up of a series of connected straight lines. It is the graph used to a large extent by business firms to show the change in expenditures, costs etc.

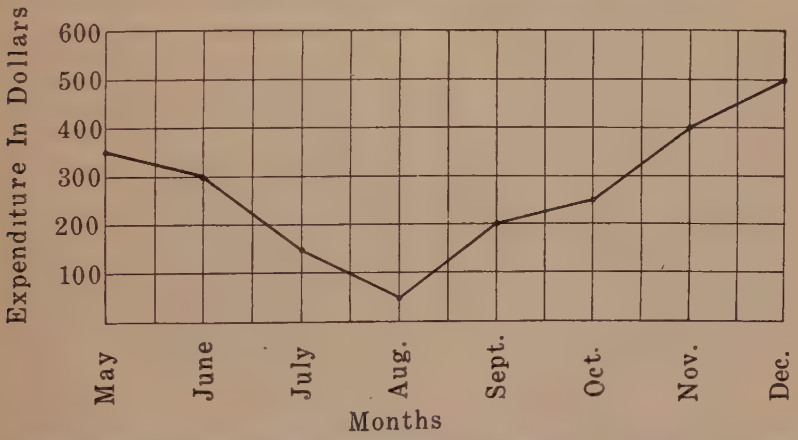


Illustration.—A firm spends a certain amount of money for advertising each month from May to December inclusive as shown by the following table.

Month....................	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Amount....................	$350	$300	$150	$50	$200	$250	$400	$500

Here the cost is summed up for an entire month and there is a distinct jump from one amount to the next during the intervals of time. This graph is made up of a series of connected straight lines.

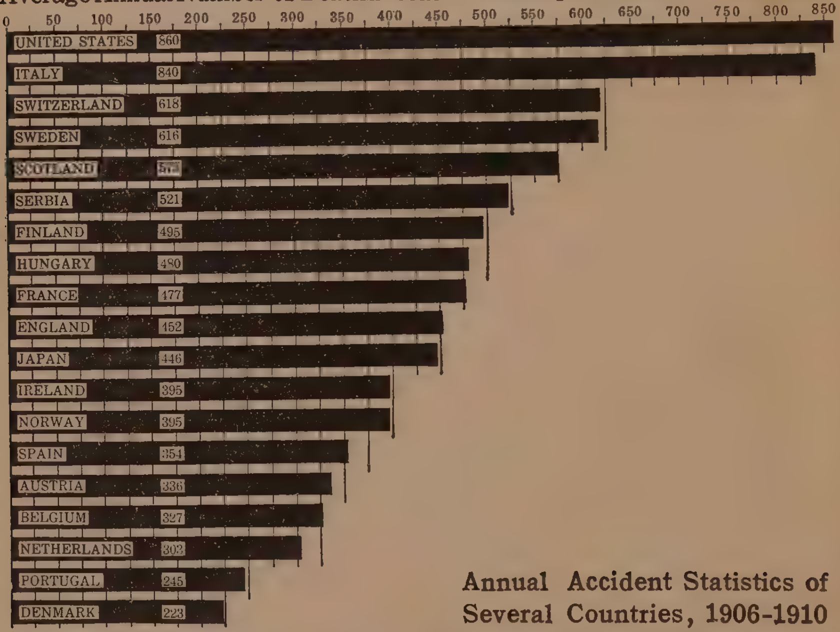
The temperature graph is a smooth curve because the change is gradual from one hour to the next.

What does each space on the horizontal axis represent? What does each space on the vertical axis represent?

Note.—*Remember that the smooth curve graph and the broken-line graph may be used to show different values of the same thing at different definite intervals.*

4. **Bar Graphs.**—An interesting variation from the above line graph

Average Annual Number of Deaths From Accidents per Million of Population

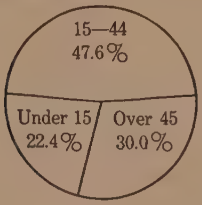


Annual Accident Statistics of Several Countries, 1906-1910

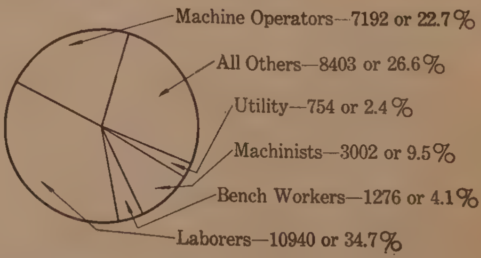
is known as the **bar graph.** This type of graph is used mainly to show relative quantities by lengths or heights of bars when the relation is in one direction only. The bars may be drawn either horizontally or vertically.

5. **Circle Graphs.**—The circle graph is used when it is desired to show the relation between several quantities and the total of which these quantities are a part. The circle divided into sectors offers the most convenient method of visualizing these relations.

Illustrations.



Fatal Accidents From All Causes
Ages at Death



Accidents by Occupation

The graphs in articles 4 and 5 are used by courtesy of The Prudential Insurance Company of America.

PROBLEMS

1. Draw a graph of the average temperature at Liverpool on the first of each month from the following data. Let each space on the vertical axis represent 1° and each space on the horizontal one month.

Jan. 1	36°	Apr. 1	46°	July 1	63°	Oct. 1	55°
Feb. 1	38°	May 1	52°	Aug. 1	64°	Nov. 1	48°
Mar. 1	42°	June 1	58°	Sept. 1	60°	Dec. 1	40°.

2. Draw a profile of a river bed from the following data. Discuss the graph.

Distance from left bank	0′,	10′,	20′,	25′,	30′,	40′,	45′,	50′,	55′,	60′,	65′.
Depth of water	5′,	9′,	15′,	18′,	23′,	25′,	27′,	32′,	27′,	20′,	10′.

3. Plot a profile from the following data:

Distance	0′,	25′,	75′,	100′,	200′,	300′,	373′,	400′,
Elevation	100′,	103.9′,	106.4′,	106.6′,	108.6′,	110.0′,	110.2′,	110.9′,
Distance	500′,	600′,	700′,	800′,	900′,	1000′,	1025′.	
Elevation	114.5′,	120.5′,	115.1′,	121.3′,	128.1′,	129.8′,	129.3′.	

Let each space equal 10 feet for horizontal distance and 1 foot for elevation.

4. Trace the pressure on a steam gauge from the following data:

Time A. M.	6,	7,	8,	9,	10,	11,	12 M.,	1 P.M.,	2,	3,	4,	5,	6.
Lbs., pressure	100,	105,	115,	125,	128,	125,	127,	125,	130,	128,	120,	122,	130.

Plot time on the horizontal axis and pressure on the vertical axis.

5. Show the comparative value of the iron and steel products of several states for one year. Values are given to the nearest millions of dollars.

Virginia	5	Wisconsin	11	Alabama	22
Tennessee	6	West Virginia	18	New York	30
Michigan	8	Indiana	19	Illinois	90
Maryland	8.5	New Jersey	22	Ohio	155
				Pennsylvania	475

Use the bar graph.

6. Represent graphically how the German U-Boats failed in six months of 1917 from the following data. Represent time on the horizontal axis and number of ships sunk on the vertical axis.

Week ending	No., ships sunk	Week ending	No., ships sunk	Week ending	No., ships sunk	Week ending	No., ships sunk
Mar. 31	31	May 26	19	July 22	24	Sept. 16	28
Apr. 7	19	June 2	18	July 29	21	Sept. 23	13
Apr. 14	28	June 9	32	Aug. 5	23	Sept. 30	13
Apr. 21	55	June 16	33	Aug. 12	16	Oct. 7	16
Apr. 28	51	June 23	28	Aug. 19	28	Oct. 14	18
May 5	47	July 1	20	Aug. 26	23	Oct. 21	25
May 12	23	July 8	17	Sept. 2	28	Oct. 28	18
May 19	27	July 15	18	Sept. 9	18	Nov. 4	12
						Nov. 11	6

7. Tension Test on Tool Steel Dia. 0.5″, Gauge Length 2″. Plot elongations on the horizontal axis and loads on the vertical axis.

Loads, lbs.	Unit elongations, inches	Loads, lbs.	Unit elongation, inches
0	0	7500	.0020
1000	.0001	8000	.0021
1500	.0002	8500	.0022
2000	.0003	9000	.0024
2500	.0004	9500	.0026
3000	.0005	10000	.0028
3500	.0007	10500	.0030
4000	.0009	11000	.0031
4500	.0011	11500	.0033
5000	.0012	12000	.0034
5500	.0014	12500	.0036
6000	.0016	13000	.0038
6500	.0017	13500	.0041
7000	.0018		

8. Plot the curve of cooling for Acetamide. Time intervals are 1 minute. Take time intervals on the horizontal axis and temperature on the vertical axis.

Time interval	Temperature	Time interval	Temperature
1	100.0°	21	66.0°
2	88.6	22	65.3
3	81.2	23	64.4
4	77.3	24	63.3
5	72.9	25	62.1
6	69.2	26	60.9
7	66.1	27	59.5
8	63.9	28	57.5
9	62.1	29	55.6
10	60.9	30	53.8
11	60.3	31	51.9
12	60.1	32	50.3
13	59.7	33	48.9
14	59.4	34	47.5
15	59.0	35	46.7
16	58.7	36	45.5
17	58.1	37	44.1
18	67.5	38	42.7
19	67.4	39	41.2
20	66.7	40	40.3

Note the peculiarity of the curve between the 17th and 18th time intervals.

9. Plot pressure on the horizontal axis and volume on the vertical axis. *P* and *V* are both given in cm. of length in a tube.

DATA SHEET OF BOYLE'S LAW

Vol. of air in cm.	Total pressure, cm., mercury	Vol. of air in cm.	Total pressure, cm., mercury
34.83	83.08	30.75	101.79
34.47	85.57	29.93	105.91
34.13	86.17	29.06	111.07
33.85	87.43	27.59	118.51
32.39	89.45	25.84	134.06
32.87	91.76	25.73	134.75
32.23	94.98	24.24	138.98
31.44	98.77		

10. Graph the following comparative statistics of building and engineering operations of Philadelphia, Baltimore and Washington as published in a Philadelphia newspaper. Values are given in millions of dollars for each month of 1918.

Jan.	60	Apr.	175	July.	265	Oct.	370
Feb.	145	May.	190	Aug.	295	Nov.	380
Mar.	165	June.	220	Sept.	340	Dec.	405

The average for 1910 to 1918 is as follows:

Jan.	15	Apr.	55	July.	90	Oct.	135
Feb.	30	May.	65	Aug.	110	Nov.	145
Mar.	40	June.	80	Sept.	120	Dec.	155

11. The gross earnings of a manufacturing firm in thousands of dollars from 1910 to 1918 inclusive are:

Year.	1910	1911	1912	1913	1914	1915	1916	1917	1918
Thousands of dollars. .	425	505	610	775	805	850	1100	1350	1550

The balances after deducting operating expenses and taxes for the same period are:

Year.	1910	1911	1912	1913	1914	1915	1916	1917	1918
Thousands of dollars. .	190	205	250	325	375	400	450	500	610

The balances after deducting interest are:

Year.	1910	1911	1912	1913	1914	1915	1916	1917	1918
Thousands of dollars. .	150	160	175	165	150	165	185	190	250

Plot the three sets of data on the same figure; discuss the value of these three graphs.

12. Measurement of the Motion of a Falling Body.

The time interval is one-twenty-seventh of a second. The velocity is given in cm. per $\frac{1}{27}$ of a second. Plot time intervals on the horizontal axis and velocity on the vertical axis.

Number of time intervals.	1,	2,	3,	4,	5,	6,	7.
Velocity at end of interval.	5.28,	6.62,	7.92,	9.30,	10.73,	12.11,	13.54.

13. Graph the purchasing value of gold based upon 22 commodities from 1860 to 1919 inclusive from the following data, compiled by the U. S. Department of Labor Bureau of Labor Statistics. 100 is the base value of the commodities.

1860	94.3	1875	79.4	1890	119.0	1905	114.9
1861	97.1	1876	85.5	1891	119.0	1906	108.7
1862	82.0	1877	87.7	1892	126.6	1907	103.1
1863	59.5	1878	97.1	1893	126.6	1908	108.7
1864	42.2	1879	101.0	1894	138.9	1909	105.3
1865	45.9	1880	90.9	1895	142.9	1910	101.0
1866	51.5	1881	93.5	1896	149.3	1911	104.2
1867	57.5	1882	91.7	1897	149.3	1912	100.0
1868	58.8	1883	97.1	1898	144.9	1913	99.0
1869	60.2	1884	107.5	1899	131.6	1914	100.0
1870	67.6	1885	114.9	1900	120.5	1915	99.0
1871	73.0	1886	117.6	1901	123.5	1916	87.7
1872	70.4	1887	117.6	1902	119.0	1917	68.5
1873	70.9	1888	113.6	1903	117.6	1918	59.5
1874	74.6	1889	117.6	1904	119.0	1919	54.5

14. In the experiment of problem 12 the distance from rest was also measured. Plot time interval on the horizontal axis and distance on the vertical axis.

Number of time intervals	1,	2,	3,	4,	5,	6,	7,	8.
Distance from rest	9.50,	15.44,	22.25,	29.89,	41.35,	52.76,	65.58,	80.85.

15. Show the following report on a circle graph. A firm expended its gross revenue as follows: 50% for salaries and wages, 20% for interest and dividends, 20% for material, rent, etc., 5% for taxes and 5% for surplus.

16. Show the following expenditures on a circle graph. The cost of a $20,000 building was as follows: Excavation and stone work $7000, iron work $1200, lumber and mill work $5000, carpenter work, etc. $4000, painting, glazing and hardware $950, heating and plumbing $1850.

17. The following is an estimate of the number of fatal industrial accidents for each 1000 male employees for one year in several of the U. S. industries.

Metal mining	3.41	Lumber industry	1.50
Coal mining	3.03	Ore dressing	1.38
Fisheries	3.00	Building and construction	1.25
Navy and Marine Corps	2.00	Ore smelting	.82
Soldiers U. S. Army	2.00	Agricultural pursuits	.35
Quarrying	1.90	Manufacturing (general)	.25

CHAPTER II

TRANSITION FROM ARITHMETIC TO ALGEBRA

6. **Numbers Represented by Letters—Literal Numbers.**—In the development of mathematics beyond arithmetic it is advantageous to represent numbers by letters. To illustrate: The area of a rectangle is equal to the length multiplied by the width. This rule was used in the study of arithmetic and was stated in words as above. It may be abbreviated, however, by representing area by the letter **A**, length by **l**, and width by **w**, and the result is a shortened statement or formula thus, **A** = **lw**. The multiplication sign is not necessary. This is the algebraic method of stating an arithmetical law.

7. **Formula.**—A formula is an expression of a rule by symbols. The symbols used in the above formula are the letters **A**, **w**, and **l** and the sign of equality. The use of letters to represent numbers is one of the distinctive features of algebra as compared with arithmetic.

Again, the interest earned by a sum of money is found by multiplying the principal by the rate per cent and that result by the time. Expressing this by letters we have the formula **i** = **p r t**, where the interest is represented by **i**, the principal by **p**, the rate per cent by **r** and the time by **t**. Thus a formula is the short-hand method of writing a mathematical rule.

Caution.—In arithmetic, the number 35 means 30 + 5, but in algebra when letters are written together as *p r t*, they indicate a product, *i.e.*, **p** times **r** times **t**.

PROBLEMS

Using the formula for the area of a rectangle **A** = **lw**, find **A** when:

1. $l = 12, \quad w = 9.$ 2. $l = 7.2, w = 9.3.$
3. $l = 52.6, w = 31.4.$ 4. $l = 31.7, w = 19.8.$

Using the formula for the area of a triangle **A** = $\frac{1}{2}$**bh**, find **A** when:

5. $b = 7, \quad h = 20.$ 6. $b = 5.5, \quad h = 37.$
7. $b = 8.1, \quad h = 17.$ 9 8. $b = 7.2, \quad h = 2.3.$
9. $b = 1.92, h = 2.4.$ 10. $b = 3.22, h = 10.5.$
11. $b = b, \quad h = a.$ 12. $b = n, \quad h = m.$
13. $b = y, \quad h = x.$ 14. $b = d, \quad h = c.$
15. $b = q, \quad h = p.$ 16. $b = t, \quad h = s.$

Using the formula **i** = **p r t**, find **i** when:

17. $p = 500, \quad r = .06, \quad t = 7.$ 20. $p = 800, \quad r = .06\frac{1}{2}, t = 3\frac{1}{2}.$
18. $p = 720, \quad r = .05, \quad t = 6.$ 21. $p = 1800, r = .04\frac{1}{4}, t = 4\frac{3}{4}.$
19. $p = 1020, r = .04\frac{1}{2}, \quad t = 5.$ 22. $p = 2100, r = .07\frac{1}{2}, t = 30.$

8. **Algebraic Expression.**—An algebraic expression is a symbol or a combination of symbols used to express a number or a quantity. Thus x^2, $a^2 + b^2$, and $2x^3 - 3x^2y + y^3$ are algebraic expressions.

ADDITION AND SUBTRACTION

9. **Addition.**—In arithmetic $16 + 5 = 21$. In algebra we can not add two different literal numbers and say that $n + r = t$ unless we know that the sum of the numerical values of **n** and **r** equals the numerical value of **t**, **n** and **r** being like units.

Illustrations.—1. If one firm has 7 trucks and a second has 8 trucks, then together they have 15 trucks. This is arithmetic addition.

2. If we say one firm has **b** trucks and a second firm has **k** trucks, then together they have b trucks + k trucks which is written $(b + k)$ trucks. This is indicated or algebraic addition.

10. **Subtraction.**—In arithmetic $25 - 9 = 16$. In algebra we cannot subtract two literal numbers except by indicating the result as $\mathbf{p} - \mathbf{q}$.

Illustrations.—1. If a man has 11 pumps and sells 7 pumps, he will have remaining 4 pumps. This is arithmetic subtraction.

2. If a firm manufactures **b** pumps one month and sells **c** pumps the same month, they have remaining in stock $(\mathbf{b} - \mathbf{c})$ pumps. This is algebraic subtraction.

Summary.—The sum of two or more literal numbers is indicated by the plus signs. The difference between two literal numbers is indicated by the minus sign.

11. **Like and Unlike Numbers.**—When we add 7 trucks and 8 trucks we get 15 trucks. If, however, we attempt to add 7 trucks and 8 wagons, which are unlike things, we cannot combine these numbers and get 15 things of the same kind. The only possible statement is 7 trucks + 8 wagons.

Similarly we cannot subtract 3 dollars from 9 chairs. The only thing we can say is 9 chairs − 3 dollars.

In both arithmetic and in algebra unlike numbers cannot actually be combined by addition or subtraction but their sum or difference is indicated by the use of the plus or minus sign.

Note.—*Problems* 17 *to* 19 *inclusive*, §7, *show that unlike numbers may be multiplied. Thus* $p\ r\ t$ *is a product of principal, rate, and time which are three unlike numbers.*

PROBLEMS

1. A floor is **h** feet long and **k** feet wide. Find its perimeter. (Perimeter equals the sum of all the sides.) Draw a figure.

2. There are **g** books in one pile and **d** books in a second pile. Find the total number of books.

3. One bin contains **v** tons of coal, a second **t** tons, a third **w** tons. How many tons are there in the three bins?

4. In problem (3), if $v = 4$, $t = 5$ and $w = 7$, how many tons would there be in all three bins?

5. A truck is carrying a load of **k** lbs. of coal, and delivers **m** lbs. at the first house. How many lbs. remain to be delivered? *Ans.* $(k - m)$ lbs.

6. If a man has **x** sheep and loses **n** sheep, how many remain?

7. A train is composed of **l** cars and **k** cars are taken off, how many remain?

Indicate:

8. What number is 6 more than **b**?
9. What number is 7 less than **k**?
10. What number is **q** more than **x**?
11. What number is 12 less than **d**?

12. **Division.**—In algebra, the process of division is the same as in arithmetic.

Illustrations.—1. Suppose 5 dozen pencils cost 50 cents. How much does one dozen cost?

In arithmetic 50 cents ÷ 5 = the cost of one dozen, or 10 cents a dozen.

In algebra we follow the same practice in the use of literal numbers.

2. If **m** books cost **t** dollars, then one book will cost **t** dollars ÷ **m** or $\frac{t}{m}$ dollars a book. Thus in division as in multiplication unlike numbers can be divided.

PROBLEMS

1. If **m** bolts weigh **n** lbs., what will one bolt weigh?
2. If **b** horses can pull a load of **t** tons, how many tons can one horse pull?
3. If a man walks **s** miles in **t** hours, how far can he walk in one hour? What is his rate per hour?
4. If the area of a rectangle is **m** square inches and the width is **r** inches. Find the length.
5. A man divides **b** dollars equally among **c** men, how many dollars will each man receive?
6. How many hours will it take a man to walk **x** miles at the rate of **y** miles per hour?

13. **Symbols Used to Indicate Multiplication.**—Several symbols are used to indicate the products of algebraic numbers.

Illustrations.—1. $b \times b = b^2$. Thus b^2 means that **b** is multiplied by **b**.

2. $3^2 = 3 \times 3$.

3. $b^3 = b \times b \times b$.

4. $3^3 = 3 \times 3 \times 3$.

5. $b^4 = b \times b \times b \times b$.

Thus, b^2 is read "b squared," and b^3, "b cubed," and b^4, "b fourth."

Then if $b = 2$, $b^3 = 2 \times 2 \times 2 = 8$ and if $c = 3$, $c^4 = 3 \times 3 \times 3 \times 3 = 81$.

The dot $\cdot$ is also used to indicate multiplication: Thus $3 \cdot 3 = 9$. Notice that the dot is written higher up than in the case of a decimal point as used in arithmetic, for example 3.3 means $3\frac{3}{10}$.

PROBLEMS

Find the value of the following:

1. x^2 if $x = 3$.
2. x^3 if $x = 2$.
3. x^4 if $x = 1$.
4. m^2 if $m = 6$.
5. n^3 if $n = 5$.
6. y^5 if $y = 2$.
7. k^4 if $k = 3$.
8. r^3 if $r = 4$.
9. s^2 if $s = 9$.
10. t^3 if $t = 6$.
11. b^4 if $b = 5$.
12. a^3 if $a = 7$.
13. c^4 if $c = m$.
14. r^3 if $r = a$.
15. t^5 if $t = b$.

14. **Factors.**—The numbers multiplied together to form a product are the factors of that product. Thus; 2 and 3 are the factors of 6; 2, 3, and 5 are the factors of 30; 2, a, a and b are the factors of $2a^2b$.

15. **Exponent.**—An exponent is the little number written to the right and above another number to indicate how many times that number is used as a factor.

The expression $2b^3$ means $2 \cdot b \cdot b \cdot b$.

Note carefully that the exponent 3 affects only the **b** and not the 2.

16. **Coefficient.**—A coefficient is a number which shows how many times an expression is added. Thus $2b^3$ means $b^3 + b^3$, 2 is the coefficient of b^3. Again, $3ab$ is a product in which 3, a, and b are the factors. Each of these factors is a coefficient of the other factors, but 3 is the numerical while **a** and **b** are the literal coefficients.

17. **Base.**—In the expression $2b^3$, **b** is called the base. Remember this picture;

$$\text{coefficient} \longrightarrow 2b^{3 \leftarrow \text{exponent}}_{\leftarrow \text{base}}$$

The expression $\dfrac{3a^2}{4}$ means $3 \times a \times a \div 4$.

Thus if $a = 2$ we have $\dfrac{3 \times 2 \times 2}{4} = 3$.

PROBLEMS

Find the value of the following:

1. ab if $a = 2$ and $b = 3$.
2. $x + y - z$ if $x = 3$, $y = 4$ and $z = 5$.
3. $m - n + p + o - q$, if $m = 12$, $n = 5$, $p = 8$, $o = 2$, $q = 10$.
4. a^2b if $a = 3$ and $b = 2$. *Ans.* 18.
5. $\dfrac{xy^2}{2}$ if $x = 5$ and $y = 4$. *Ans.* 40.
6. $a \cdot a \cdot b$ if $a = 5$ and $b = 4$.
7. $\dfrac{a+b}{r-s}$ if $a = 5$, $b = 9$, $r = 7$, $s = 5$.
8. $3bc - 2kr$ if $b = 7$, $c = 2$, $k = 3$, $r = 2$.
9. $4m^2n^2$ if $m = 2$, $n = 3$.
10. $\dfrac{a+b}{3x} + \dfrac{r-s}{t+w}$ if $a = r$, $b = 4$, $x = 3$, $r = 10$, $s = 8$, $t = 2$, $w = 1$. *Ans.* $2\frac{2}{9}$.

18. **Laws of Exponents and Coefficients.**—1. When the bases are the same add the exponents of the factors to obtain the exponent of the product.

Thus $a^3 \times a^5 = a^8$ and $m^7 \times m^3 = m^{10}$.

2. The coefficient of the product is equal to the product of the coefficients of its factors. Thus, $2a^2 \times 3a^3 = 6a^5$.

3. The product of two or more numbers involving several bases, is equal to the product of all the numerical coefficients and all the literal factors found in the different terms, each base having for its exponent the sum of the exponents of that base in the several factors. Thus, $2a^2b^3 \times 5a^3b^5 \times 3ab^2c = 30a^6b^{10}c$.

PROBLEMS

Find the product of the following:

1. $x^2y \cdot xy^2$.
2. $3ab^2 \times 2a^2b^3$.
3 $m^2n^3 \times mnp$.
4. $3rst^2 \times 5r^3sp^2$.
5. $6m^2o \times 7m^2o^3$.
6. $5abc \times 6abcd$.
7. $10xyz \times 12z^2$.
8. $9bcd \times 3a$.
9. $5ac \times 2bc$.
10. $6fg \times 7frw$.
11. $5p^2q \cdot 3pq^2$.
12. $7a^2b \cdot cd^2$.

19. **Unit Exponent.**—When the exponent is not expressed the unit 1 is understood. Thus **a** means **a** to the first power and $a \times a = a^2$ means $a^1 \times a^1 = a^2$. The law of exponents is seen to apply here since the sum of the exponents 1 and 1 equals 2.

20. **Parentheses.**—Parentheses, ( ) are used as symbols of grouping to indicate that all the numbers inclosed by them are to be treated as a single number. Thus $3(4 + 5)$ means that the sum of 4 and 5 is to be multiplied by 3; $(6 - 2)(4 + 1)$ means 4 multiplied by 5; $(a + b)(c + d)$ means that the sum of **a** and **b** is to be multiplied by the sum of **c** and **d**.

PROBLEMS

Find the value of the following:

1. $2(4 + 6)$.
2. $3(6 + 5)$.
3. $4(5 + 2)$.
4. $7(9 - 2)$.
5. $\frac{1}{2}(6 + 4)$.
6. $\frac{1}{3}(12 - 3)$.
7. $\frac{1}{4}(9 - 7)$.
8. $3(3 + 5 + 8)$.
9. $6(2 + 3 - 1)$.
10. $7(9 + 2 - 8)$.
11. $3(x + y^2 + z^3)$ when $x = 2$, $y = 3$, $z = 4$.
12. $4(m^2 - n^2 + p)$ when $m = 5$, $n = 3$, $p = 8$.
13. $5(r + s^2 - v)$ when $r = 6$, $s = 5$, $v = 9$.
14. $10(k^2 - l^2 + p + t)$ when $k = 5$, $l = 3$, $p = 7$, $t = 9$.
15. $\dfrac{6(a + b + d)}{2}$ when $a = 3$, $b = 4$, $d = 7$.
16. $\dfrac{5(d + e^3 + f^2)}{4}$ when $d = 7$, $e = 2$, $f = 3$.
17. $p^2(p + o + r)$ when $p = 3$, $o = 4$, $r = 6$.
18. $x^3(x - c + d)$ when $x = 6$, $c = 2$, $d = 5$.
19. $r(b + k^2 + r)$ when $r = 2$, $b = 7$, $k = 10$.
20. $k(k^2 + s + a)$ when $k = 4$, $s = 2$, $a = 6$.
21. $a(a^2 - b + c)$ when $a = 2$, $b = 3$, $c = 5$.
22. $m^2(m^2 - n + p)$ when $m = 3$, $n = 8$, $p = 9$.

21. **Terms.**—The parts of an algebraic expression which are separated by + or − signs are the terms of the expression. Thus $3a + 5b - 7c$ is an expression of three terms; $3a$, $5b$, and $7c$ are the terms.

22. **Monomial.**—A monomial is an algebraic expression of one term, as x, $2a$, $3b$, $5d$.

23. **Binomial.**—A binomial is an algebraic expression of two terms, as $2 - a$; $3 + b$; $2x - 3y$.

24. **Trinomial.**—A trinomial is an algebraic expression of three terms; as $2a - b + 3c$; $4x + 5y - 9z$.

25. **Polynomial.**—A polynomial is a general name for an algebraic expression of two or more terms.

ORAL PROBLEMS

Express in algebraic form and tell the kind of algebraic expression.

1. The sum of the squares of **x** and **y**.
2. The difference of the cubes of **p** and **r**. (Two results.)
3. The sum of the square of **a** and the cube of **y**.
4. The difference of the cube of **k** and **m**.
5. One-half of the sum of **a** and **2b**.
6. The square of the sum of **a** and **b**. Thus $(a + b)^2$.
7. The square of the difference of **x** and **v**.
8. One-third of the product of **4a** and **x**.
9. The cube of the sum of **m** and **n**.
10. The sum of **x** and **y** diminished by **z**.
11. The number that is **p** less than **q**.
12. The number that is **8** more than **d**.
13. The result of adding **r** times **s** to **t**.
14. The difference of the squares of **mn** and **pq**.
15. If **a, b,** and **c** are the sides of a triangle, find the perimeter. (Draw figure.)
16. If l is the length and w the width of a room, find the distance around. (Draw figure.)

POSITIVE AND NEGATIVE NUMBERS—SIGNED NUMBERS

26. **Symbols.**—In arithmetic the symbols + and − indicate addition and subtraction only. We will now use them in algebra for still another purpose without losing any of their value as signs of addition and subtraction. In algebra we speak of "positive and negative numbers." A term preceded by a plus sign, as $(+3)$ or $(+b)$, is a positive number. If preceded by a minus sign, as (-4) or $(-r)$, it is a negative number.

Read the following and tell which are positive and which are negative: $(+5)$, (-7), (-11), $(+27)$, $(+0)$, $(-k)$, $(-s)$. Notice that as we proceed we will meet with terms which have neither a plus nor a minus sign preceding them. In that case we agree to consider the plus sign as understood. That is **b** is a positive number just the same as if it were written **+b**. If a number is negative it must always be preceded by a minus sign.

Tell which of the following are negative and which positive: $(+a)$, (b), $(-c)$, (k), $(+f)$, $(-m)$, (-0), (p), $(-r)$, $(-n)$, $(+q)$, (t).

From the above list make two columns, one containing only positive and the other only negative numbers. Now you have classified them in groups having like signs.

27. **Absolute Value.**—The absolute value of a number is the value of that number without regard to the sign. Thus $|a|$ means that the sign of **a** is not considered.

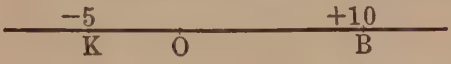
28. **Quality of Numbers.**—The terms positive and negative are used to show the opposite nature or quality of numbers.

Illustrations.—1. We usually say the temperature is 6 degrees above zero. We could say the temperature is +6°. If the temperature is 3°

below zero we could say −3°. Notice that the temperature must rise 9° from −3° to +6°.

2. Consider now, money on hand as against money owed. A man has $50 in bank, that is +$50 or assets and he owes $20 to his creditors that is −$20 or liabilities. Here the opposite nature is again shown and when he pays his indebtedness he will have only $30 in bank, *i.e.*, +$30.

3. We speak of 10 miles east as being +10 miles and 15 miles west as −15 miles. If we start at point O, and travel east to point B, +10 miles and turn in the opposite direction then travel west to point K, −15 miles from B, we would have reached the same point by traveling −5 miles (*i.e.*, west from O) in the first place.



29. **Addition of Signed Numbers. Illustrations.**—Money gained is designated (+) and money lost (−).

1. Find the sum of +8 and +7, *i.e.*, a gain of 8 plus a gain of 7 = a gain of 15. ∴ (+8) + (+7) = +15.

2. Find the sum of −5 and −9, *i.e.*, a loss of 5 plus a loss of 9 = a loss of 14. ∴ (−5) + (−9) = −14.

3. Find the sum of (−7) and (+2), *i.e.*, a loss of 7 plus a gain of 2 is equivalent to a net loss of 5. ∴ (−7) + (+2) = −5.

4. Find the sum (+7) and (−2), *i.e.*, a gain of 7 plus a loss of 2 is equivalent to a net gain of 5. ∴ (+7) + (−2) = +5.

Hence, the rule for adding signed numbers is in two parts, one to apply when the signs are alike, *i.e.*, both minus or both plus and the other when one sign is plus and the other minus. Therefore:

1. *When the signs are alike add as in arithmetic and prefix the common sign to the result.*

2. *When the signs are unlike subtract the smaller number from the larger as in arithmetic and prefix the sign of the larger number.*

Test these rules on the above illustrations.

PROBLEMS

Add the following:

1. +2 +7	2. + 9 +11	3. −8 −6	4. −9 −7	5. +3 −1	6. +5 −9	7. +8 −5
8. +21 −22	9. −4 +4	10. − 9 −10	11. +12 +31	12. +18 −10	13. +7 −9	14. −9 +7
15. −10 −11	16. + 9 −12	17. −12 + 9	18. +18 − 6	19. − 6 +18	20. $9p$ $-7p$	21. $3x$ $-5x$
22. $-8r$ $-6r$	23. $12t$ $-12t$	24. $8m$ $-8m$	25. $9b$ $6b$	26. $101v$ $-82v$	27. $-230n$ $118n$	
28. $-121c$ $-111c$	29. $89d$ $-89d$	30. $125ab$ $-78ab$	31. $-81mn$ $22mn$	32. $119k$ $-121k$	33. $-25pq$ $-79pq$	

Note.—*Before going further these simple problems in addition must be thoroughly mastered.*

30. **Addition of Three or More Terms.**—To add a column containing three or more similar terms with unlike signs, first find the sum of the positive terms and negative terms separately and then add their results.

Illustration.

$$\begin{array}{rl} 3ab & \left\{\begin{array}{l}\text{The sum of } 3ab \text{ and } 10ab \text{ is } 13ab. \\ \text{Now add } -4ab \text{ to } 13ab \text{ and we have } 9ab.\end{array}\right\} \\ -\ 4ab & \\ 10ab & \\ \hline 9ab & \text{Result.}\end{array}$$

31. **Addition of Polynomials.**—To add polynomial expressions, first arrange similar terms in vertical columns and proceed as in §30.

Illustrations.—1. Add $12x - 15y + 7z$, $-5x + 4y - 2z$ and $-3x + 10y - 14z$.

Solution.

$$\begin{array}{rrr} 12x & -\ 15y & +\ \ 7z \\ -\ 5x & +\ \ 4y & -\ \ 2z \\ -\ 3x & +\ 10y & -\ 14z \\ \hline 4x & -\ \ \ y & -\ \ 9z \end{array} \text{ Result.}$$

2. Add $ax + by$, $bx - cy$, and $cx + dy$.
Check if $a = 1$, $b = 2$, $c = 3$, $d = 4$, $x = 5$ and $y = 2$.

Solution. **Check**

$$\begin{array}{c} ax + by \\ bx - cy \\ cx + dy \\ \hline (a + b + c)x + (b - c + d)y \end{array} \qquad \begin{array}{r} 1{\cdot}5 + 2{\cdot}2 = \ \ 9 \\ 2{\cdot}5 - 3{\cdot}2 = \ \ 4 \\ 3{\cdot}5 + 4{\cdot}2 = 23 \\ \hline (1 + 2 + 3)5 + (2 - 3 + 4)2 = 36 \end{array}$$

In (2) we add the coefficients of x and y as before, but since they are literal numbers their sum can only be indicated. The parentheses must be used to show that the sum of **a, b,** and **c** is the coefficient of **x** and that the sum of **b**, **−c**, and **d** is the coefficient of **y**. To write **a + b + cx** means that **c** alone is the coefficient of **x**.

PROBLEMS

Find the sum of the following, check results. Do not use a value of 1 for any letter raised to a power. It will not check the exponent.

1. $\begin{array}{r} 3x \\ 5x \\ 8x \\ \hline \end{array}$

2. $\begin{array}{r} -7m \\ 2m \\ -3m \\ \hline \end{array}$

3. $\begin{array}{r} 3xy^2 \\ 6xy^2 \\ -8xy^2 \\ \hline \end{array}$

4. $\begin{array}{r} -7x^2y \\ 8x^2y \\ -2x^2y \\ \hline \end{array}$

5. $\begin{array}{r} 8pq \\ -\ pq \\ +\ pq \\ \hline \end{array}$

6. $\begin{array}{r} m^2n \\ 7m^2n \\ -8m^2n \\ \hline \end{array}$

7. $\begin{array}{r} 3x^2y^2z \\ 7x^2y^2z \\ -10x^2y^2z \\ \hline \end{array}$

8. $\begin{array}{r} 3rst \\ 5rst \\ -8rst \\ \hline \end{array}$

9. $\begin{array}{r} 2(x - y) \\ 3(x - y) \\ -6(x - y) \\ \hline \end{array}$

10. $\begin{array}{r} 4(a^2 - b) \\ 5(a^2 - b) \\ -7(a^2 - b) \\ \hline \end{array}$

11. $$\begin{array}{r} 10(x - y + z) \\ 11(x - y + z) \\ -20(x - y + z) \\ \hline \end{array}$$

12. $$\begin{array}{r} -5(m - n)^2 \\ 6(m - n)^2 \\ -12(m - n)^2 \\ \hline \end{array}$$

13. $$\begin{array}{r} ax \\ bx \\ \hline \end{array}$$

14. $$\begin{array}{r} ax \\ -bx \\ \hline \end{array}$$

15. $$\begin{array}{r} a(x + y) \\ b(x + y) \\ \hline \end{array}$$

16. $$\begin{array}{r} arx \\ ary \\ \hline \end{array}$$

17. $$\begin{array}{r} ab \\ -b \\ \hline \end{array}$$

18. $$\begin{array}{r} km \\ pm \\ \hline \end{array}$$

19. $$\begin{array}{r} ry \\ -sy \\ \hline \end{array}$$

20. $$\begin{array}{r} 5x \\ ax \\ \hline \end{array}$$

21. $$\begin{array}{r} bd \\ -9d \\ \hline \end{array}$$

22. $$\begin{array}{r} -5ax \\ bax \\ \hline \end{array}$$

23. $$\begin{array}{r} my \\ -3y \\ \hline \end{array}$$

24. $$\begin{array}{r} rs \\ ps \\ \hline \end{array}$$

25. $$\begin{array}{r} m(p + q) \\ -n(p + q) \\ \hline \end{array}$$

26. $$\begin{array}{r} 3a - 5b \\ 4a + 7b \\ \hline \end{array}$$

27. $$\begin{array}{r} 8x - 9y \\ 3x + 2y \\ \hline \end{array}$$

28. $$\begin{array}{r} 2ab + b^2 \\ -3ab - 2b^2 \\ \hline \end{array}$$

29. $$\begin{array}{r} a^2b - 2ab^2 \\ 3a^2b - 7ab^2 \\ \hline \end{array}$$

30. $$\begin{array}{r} a^3b^2 + 5ab^2 + 6ab^3 \\ 7a^3b^2 + 2ab^2 - 9ab^3 \\ \hline \end{array}$$

31. $$\begin{array}{r} 8mnp - 3mnq \\ -6mnp + 10mnq \\ \hline \end{array}$$

32. $$\begin{array}{r} a + b - c + d \\ 2a - 3b + c + d \\ \hline \end{array}$$

33. $$\begin{array}{r} \frac{1}{2}a + \frac{1}{2}c \\ \frac{1}{2}a - \frac{1}{2}c \\ \hline \end{array}$$

34. $$\begin{array}{l} m - 2n \\ m \qquad + 3 \\ \hline \end{array}$$

35. $3x - 2y + 4z$, $2x + y - 3z$, $-x + 3y - z$.
36. $2a + 3b - 7c$, $-3a - 2b + 5c$, $6a - 2b + c$.
37. $3a^2 - 7b^2 - 7c$, $2a^2 - b^2 - c$, $5a^2 + 2b^2 - 3c$.
38. $3x^2 - y^2 + 2xy$, $4xy - 7x^2 + y^2$, $y^2 - xy + 4x^2$.
39. $2m - 3n^2 + mn$, $3m - mn - 2n^2$, $4n^2 - 3m + mn$.

32. **Associative Law for Addition.**—The sum of numbers grouped together is the same as the sum of the same numbers when the symbols of grouping are ignored. This is known as the "associative law for addition." Thus; $2 + 3 + 1 = 2 + (3 + 1) = (2 + 3) + 1 = 6$. $\therefore x + (y + z) = (x + y) + z$, etc.

33. **Commutative Law for Addition.**—The sum of numbers, positive or negative is not affected by a change in the order of the numbers. This is known as the "commutative law for addition." $5 + 2 = 2 + 5$ also $a + b + c = b + a + c$, etc.

34. **Subtraction of Signed Numbers. Illustrations.**—1. In arithmetic when not thinking of positive or negative numbers if we add 7 and 2 we get 9. Then it follows that the 2 subtracted from the 9 must give the 7 back again. Thus: $(+7) + (+2) = +9$ and $(+9) - (+2) = +7$.

2. The same must be true in the case of negative numbers, for $(+9) + (-5) = +4$. Now if we subtract (-5) from $(+4)$ we must get 9 back again. Thus: $(+4) - (-5) = +9$.

Therefore, in subtraction of signed numbers think of the sign of the lower number, called the subtrahend, as being changed and proceed as in addition.

ORAL PROBLEMS

Subtract the following:

1. $$\begin{array}{r} +7 \\ +2 \\ \hline \end{array}$$

2. $$\begin{array}{r} -7 \\ -2 \\ \hline \end{array}$$

3. $$\begin{array}{r} -7 \\ +2 \\ \hline \end{array}$$

4. $$\begin{array}{r} +7 \\ -9 \\ \hline \end{array}$$

5. $$\begin{array}{r} +7 \\ -7 \\ \hline \end{array}$$

6. $$\begin{array}{r} -8 \\ -9 \\ \hline \end{array}$$

7. $\begin{array}{r}+8\\+9\\\hline\end{array}$ 8. $\begin{array}{r}+9\\-17\\\hline\end{array}$ 9. $\begin{array}{r}+17\\-9\\\hline\end{array}$ 10. $\begin{array}{r}-17\\+9\\\hline\end{array}$ 11. $\begin{array}{r}-8\\+8\\\hline\end{array}$ 12. $\begin{array}{r}+12\\-12\\\hline\end{array}$

13. $\begin{array}{r}-9\\+11\\\hline\end{array}$ 14. $\begin{array}{r}+9\\+11\\\hline\end{array}$ 15. $\begin{array}{r}-21\\+21\\\hline\end{array}$ 16. $\begin{array}{r}7x\\-5x\\\hline\end{array}$ 17. $\begin{array}{r}-8m\\-7m\\\hline\end{array}$ 18. $\begin{array}{r}12r\\10r\\\hline\end{array}$

19. $\begin{array}{r}10r\\12r\\\hline\end{array}$ 20. $\begin{array}{r}-15s\\+15s\\\hline\end{array}$ 21. $\begin{array}{r}+15s\\-15s\\\hline\end{array}$ 22. $\begin{array}{r}+16k\\+16k\\\hline\end{array}$ 23. $\begin{array}{r}-20x\\+22x\\\hline\end{array}$ 24. $\begin{array}{r}25b\\-17b\\\hline\end{array}$

25. $\begin{array}{r}-39y\\+18y\\\hline\end{array}$ 26. $\begin{array}{r}-39y\\39y\\\hline\end{array}$ 27. $\begin{array}{r}-39y\\-39y\\\hline\end{array}$ 28. $\begin{array}{r}7a+3b\\-2a-\ b\\\hline\end{array}$

29. $\begin{array}{r}10x-9y\\-21x+8y\\\hline\end{array}$ 30. $\begin{array}{r}8m-6p\\-12m\mp 3p\\\hline\end{array}$ 31. $\begin{array}{r}12t+7v\\-18t+10v\\\hline\end{array}$ 32. $\begin{array}{r}9k-10q\\-10k+21q\\\hline\end{array}$

33. $\begin{array}{r}19b-12c\\-18b+2c\\\hline\end{array}$ 34. $\begin{array}{r}5r-9a\\-16r+12a\\\hline\end{array}$ 35. $\begin{array}{r}3ab+2mc\\+9ab+6mc\\\hline\end{array}$ 36. $\begin{array}{r}-10a^2d-21f^2g\\-15a^2d+18f^2g\\\hline\end{array}$

37. $\begin{array}{r}6xy+9r^3s\\+10xy+10r^3s\\\hline\end{array}$ 38. $\begin{array}{r}22k^2m-9pr\\-16k^2m+21pr\\\hline\end{array}$ 39. $\begin{array}{r}-13xyz^3+12kr^4z\\-22xyz^3+10kr^4z\\\hline\end{array}$

40. $\begin{array}{r}18cd^4+22rk^2\\-19cd^4+22rk^2\\\hline\end{array}$ 41. $\begin{array}{r}-35m^2p^2-10w^3z^3\\-35m^2p^2+10w^3z^3\\\hline\end{array}$ 42. $\begin{array}{r}108l^2m-19p^4q^3\\-91l^2m+19p^4q^3\\\hline\end{array}$

43. $\begin{array}{r}113z^2m^4-18pq^2\\-113z^2m^4+18pq^2\\\hline\end{array}$ 44. $\begin{array}{r}ax\\-bx\\\hline\end{array}$ 45. $\begin{array}{r}ax\\-bx\\\hline\end{array}$ 46. $\begin{array}{r}-cy\\dy\\\hline\end{array}$

47. $\begin{array}{r}mr\\2r\\\hline\end{array}$ 48. $\begin{array}{r}3k\\pk\\\hline\end{array}$ 49. $\begin{array}{r}axy\\bxy\\\hline\end{array}$ 50. $\begin{array}{r}m(x+y)\\n(x+y)\\\hline\end{array}$

35. Symbols of Grouping.—When it is desired to treat a group of two or more terms as one term we use symbols of grouping to show it: Thus; $5-(3+1)=1$. Here it is understood that we are to add 3 and 1 first which gives 4 and subtract the result from 5. Thus $5-(3+1)=5-4=1$.

The symbols of grouping used are the following: Parentheses ( ), braces { }, brackets [ ] and vinculum $\overline{\qquad}$

Illustrations.—1. $8+(3+7)$ means that the sum of 3 and 7 is to be added to 8. We have $8+10=18$. If we write $8+3+7$ the result is also 18. In general $a+(b+c)=a+b+c$ and $2a+(3b-2c)=2a+3b-2c$.

Therefore; (*a*) *If an expression within parentheses is preceded by a plus sign, the parentheses may be removed without changing the sign of any of the terms within.*

(*b*) *It follows from* (*a*) *that any number of terms of an expression may be enclosed within parentheses preceded by a plus sign without changing the signs of the terms enclosed within the parentheses.*

2. $9-(3+2)$ means that the sum of 3 and 2 is to be subtracted from 9. The result is 4. Since in subtraction we change the sign of the

number to be subtracted and proceed as in addition, we write $9 - 3 - 2$. The result is 4 as above.

Note.—*The minus sign before the parentheses is a sign of operation and is dropped after the signs of the terms within the parentheses have been changed. In general* $a - (b + c) = a - b - c$ *and* $2a - (3b - 2c) = 2a - 3b + 2c$.

Therefore: (*a*) *If an expression within parentheses is preceded by a minus sign, the parentheses may be removed only if the sign of every term within the parentheses is changed.*

(*b*) *It follows from* (*a*) *that any number of terms of an expression may be enclosed in parentheses preceded by a minus sign only if the sign of each term enclosed is changed.*

When symbols of grouping occur within other symbols of grouping, the innermost symbols should be removed first.

Illustration.—Simplify $-[2a - (3b + \overline{a - 2b})]$

Solution.

$$\begin{aligned} & -[2a - (3b + a - 2b)] \\ = & -[2a - 3b - a + 2b] \\ = & -2a + 3b + a - 2b \text{ or } b - a. \end{aligned}$$

PROBLEMS

Simplify the following:

1. $8 + (6 - 1)$.
2. $3a + (2a - a)$.
3. $9 - (7 - 2)$.
4. $x - (y - 2x)$.
5. $7 + (3 + 4)$.
6. $b + \overline{2b - b}$.
7. $9 - [2 + 4]$.
8. $4r - [2r + 3]$.
9. $7 + 3 - 1$.
10. $k + 2k + 3k$.
11. $8 - (3 + a) + (6 - 1)$.
12. $a - (2a + 1) + (3a - 2)$.
13. $12 - (-4 + 7) + (8 - 2)$.
14. $m - (-2m + 3) + (5m - 6)$.
15. $x - (x + y) + (2x - 3y)$.
16. $2y - (3x + 2y) + (4x - y)$.
17. $b + (2b - [b + 4] - 5)$.
18. $p + 3p - \{2p + \overline{4 - 5p} + 6\}$.
19. $4c - [-2c + (3c - 7) + 8c]$.
20. $2x - [-(3x + 2y) + 6]$.
21. $c + \{-3c - [c - (b + 2c) + b] + 8c\}$.
22. $3x - [2y - (x + y) - 3y]$.
23. $3m - 2n + \{m - (2m + 3n)\}$.
24. $-(3p - 2q) - [q - (3p + q) - 3p]$.
25. $14 - [10 + (9 - 8 + \overline{7 + 2}) - 1]$.
26. $5 - [-5 + (5 - 2) + (-3) - 1]$.
27. $m - [m - (n + r) - \{m - (n + r)\}]$.
28. $2 + \{-2 + [3 - (1 - \overline{2 + 1}) - 2] + 1\}$.
29. $(x + 2y + 3z) - (2x - y - 2z) + (z - x - y)$.
30. $1 - [-1 + (1 - 1 + \overline{1 - 1}) - (1 - 1)] - 1$.

In the following polynomials enclose the last three terms within parentheses preceded by a plus sign.

31. $a - b + c - d$.
32. $a + b - c + d$.
33. $x + 2y - 3k + 2d$.
34. $9 - 3b + 2c - 8d$.
35. $m - 2n + p + 2$.
36. $r + 2s - 3t + 4u$.
37. $2c - 3d - 4e + f$.
38. $12 - 10 + 5 - 7$.
39. $m - n + p - q$.

Enclose the last three terms of each of the expressions 31 – 39 within parentheses preceded by a minus sign.

36. **Multiplication of Signed Numbers.**—Multiplication in arithmetic is a short method of addition. It is the process of taking one number as many times as there are units in another. Now, we must consider the

quality as well as the quantity of the numbers used. The definition then means that we use the multiplicand in the same way as we use unity (1) to obtain the multiplier.

Illustrations.—1. Find 3 times 6.

Solution. To obtain 3 by using 1, we have, $0 + 1 + 1 + 1 = 3$. Hence $0 + 6 + 6 + 6 = +18$.

2. Find 3 times (-6).

Solution. Following the method of (1) we have $0 + (-6) + (-6) + (-6) = 0 - 6 - 6 - 6 = -18$.

3. Find (-3) times (6).

Solution.—To obtain -3 we have $0 - (1) - (1) - (1) = 0 - 1 - 1 - 1 = -3$.

Hence, $0 - (+6) - (+6) - (+6) = 0 - 6 - 6 - 6 = -18$. §35–2*a*.

4. Find (-3) time (-6).

Solution. Following the method of (3) we have $0 - (-6) - (-6) - (-6) = 0 + 6 + 6 + 6 = 18$. §35–2*a*.

37. Law of Signs for Multiplication.

1. *The product of two numbers having like signs is plus* (+).

2. *The product of two numbers having unlike signs is minus* (−).

3. *The sign of the product of three or more numbers is found by multiplying the product of any two numbers by a third number and so on.*

38. Summary for Multiplication.

1. *Like signs give plus, unlike signs give minus.*

2. *Multiply coefficients.*

3. *Add exponents of the same factors.*

PROBLEMS

Find the value of the following:

1. $(+3)(+2)$.
2. $(-3)(-4)$.
3. $(-6)(+3)$.
4. $(-3)(+6)$.
5. $(+10)(-2\frac{1}{2})$.
6. $(+3)(+4)(+5)$.
7. $(+2)(+4)(-2)$
8. $(-6)(-2)(-3)$.
9. $(+6)(+2)(-3)$.
10. $(+9)(-2)(+3)$.
11. $(-16)(+\frac{1}{2})$.
12. $(+18)(-\frac{1}{3})$.
13. $(+20)(-5)(-\frac{1}{4})$.
14. $(-2)(-2)(-2)(-2)$.
15. $(-2)(+3)(-4)(+5)$.
16. $(7a^2)(9a^3)$.
17. $(-8x^2)(10xy)$.
18. $(-6ab^2)(7abc)$.
19. $(-3rs)(5rs^2)$.
20. $(mx^3)(-9m^2x)$.
21. $(-14ax^2)(-8axy)$.
22. $(-15am)(-7ma)$.
23. $(-3x^3y^3z^3)(-9x^2yz^4)$.
24. $(18abc)(9bcd^2)$.
25. $(-105my)(-3xmy^2)$.

39. Commutative Law for Multiplication.—The product of two or more numbers is not affected by a change in the order of the numbers. Thus, $5 \cdot 3 = 3 \cdot 5$ and $a \cdot b = b \cdot a$. This is known as the **commutative law for multiplication.**

40. Associative Law for Multiplication.—The product of numbers is not affected by the order of grouping. Thus: $2(3 \times 5) = (2 \times 3)5$ and $a(b \times c) = (a \times c)b$. This is known as the **associative law for multiplication.**

41. Multiplication of a Polynomial by a Monomial.—To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and write the resulting products in succession prefixing the proper sign before each term.

Illustration.—$2(4 + 5 - 3) = 2 \cdot 6$ or 12.

Also $2 \cdot 4 + 2 \cdot 5 - 2 \cdot 3 = 8 + 10 - 6$ or 12.

Thus $a(b + c - d) = ab + ac - ad$.

The above illustrates what is known as the **distributive law for multiplication.**

PROBLEMS

Find the value of the following:

1. $5(a - b)$.
2. $6(x + 2y)$.
3. $8a(a - b)$.
4. $-2x(x + x^2)$.
5. $3y(y^2 + 1)$.
6. $2x(a + b)$.
7. $-a(b - a)$.
8. $4c(-d - 2c)$.
9. $3r(-2r + r^2)$.
10. $3r^2(-2r + r^2)$.
11. $-2b(b - c + d)$.
12. $3a(a + b - c + d)$.
13. $4k(-r + t - v)$.
14. $-8c(-c^2 - d + e - f)$.
15. $2b^2(c - d^2 + e)$.
16. $4(a - 3b) + 2(a + 2b) - 5(6a - b)$.

Solution. $= 4a - 12b + 2a + 4b - 30a + 5b$
$= -24a - 3b$.

17. $3(x - 2y) - 2(x + 3y) + (x - y)$.
18. $4(a - 6b) + 3(-2a + b) - 2(3a - 2b)$.
19. $6(3a + 2c) - 4(a - c) + 2(a - 3c)$.
20. $2a(b + d) - 4a(2b + 3d) + a(b + d)$.
21. $a(b + y) - 2a(b - 2y) + 3a(2b - y)$.
22. $3m(m + n) - 2m(2m - n) + m(2m - n)$.

42. **Multiplication of a Polynomial by a Polynomial.**

Illustrations.

1.
$$\begin{array}{ll} a+b & \\ c+d & \\ \hline ac+bc & \\ & ad+bd \\ \hline \multicolumn{2}{l}{ac+bc+ad+bd} \\ \multicolumn{2}{l}{8+12+10+15=45} \end{array}$$

Let $a = 2$, $b = 3$
$c = 4$, $d = 5$

$$\begin{array}{ll} 2+3 & \\ 4+5 & \\ \hline 8+12 & \\ & 10+15 \\ \hline \multicolumn{2}{l}{8+12+10+15=45} \end{array}$$

Study the two problems in multiplication. Notice that the operation with literal numbers is identically the same as with the arithmetical numbers and that when we substitute numerical values for letters we get a corresponding product 45 for both, thus checking our work.

The operation is the same in all multiplication.

2.
$$\begin{array}{ll} 3a + b & \\ 2a - 3b & \\ \hline 6a^2 + 2ab & \\ \quad\;\; - 9ab - 3b^2 & \\ \hline 6a^2 - 7ab - 3b^2 & \end{array}$$

Check if $a = 2$
and $b = 3$

$$\begin{array}{rcr} \text{Then } 3a + b & = & 9 \\ 2a - 3b & = & -5 \\ \hline 6a^2 - 7ab - 3b^2 & = & -45 \\ -45 & = & -45 \end{array}$$

PROBLEMS

Find the value of the following: Check.

1. $(x+y)(x+y)$.
2. $(x+2)(x+2)$.
3. $(x+3)(x-4)$.
4. $(2a+3b)(a+b)$.
5. $(2a-3b)(a+b)$.
6. $(2a-3b)(a-b)$.
7. $(4x+y)(3x-2y)$.
8. $(8m+n)(2m+n)$.
9. $(x+2y)(x-2y)$.
10. $(3x+y)(3x-y)$.
11. $(4a+b)(4a+b)$.
12. $(2r-6)(r-4)$.
13. $(x-6y)(3x+2y)$.
14. $(2r+3)(4r-5)$.
15. $(a+b)(c+d)$.
16. $(m+n)(r+s)$.
17. $(b+g)(d-e)$.
18. $(f+k)(r-s)$.
19. $(2e+d)(a-c)$.
20. $(l-m)(3l+2)$.
21. $(x^2-2)(x-3)$.
22. $(x^2-6)(x+4)$.
23. $(x^3-2)(x^2-4)$.
24. $(x^3-2)(x^2+4)$.
25. $(x^2y-6)(x^2y+3)$.
26. $(ab^2-c)(2ab^2+3c)$.
27. $(m^3-n)(m^2+n)$.
28. $(m^2+2n)(m^2-2n)$.
29. $(3k+71)(6k-81)$.
30. $(9r-7s)(10r-13s)$.

43. **Division of Signed Numbers.**—Division is the inverse of multiplication.

Illustrations.

1. $2(-3) = -6 \therefore -6 \div -3 = 2$. Also $-6 \div 2 = -3$.
2. $a^3 \cdot a^2 = a^5 \quad \therefore a^5 \div a^3 = a^2$. Also $a^5 \div a^2 = a^3$.
3. $3a^4 \cdot 2a^3 = 6a^7 \therefore 6a^7 \div 3a^4 = 2a^3$. Also $6a^7 \div 2a^3 = 3a^4$.

Hence, the rules for division may be stated as follows:

1. *Like signs give plus, unlike signs give minus.*
2. *Divide coefficients.*
3. *Subtract the exponent of each factor in the divisor from the exponent of the same factor in the dividend.*

PROBLEMS

Solve the following: Check as in §31.

1. $35 \div 7$.
2. $42 \div 6$.
3. $18 \div -3$.
4. $-21 \div 3$.
5. $-21 \div -3$.
6. $-24 \div -6$.
7. $-18a \div 3$.
8. $18a \div -3$.
9. $-8 \div -1$.
10. $-9 \div -1$.
11. $3ab \div 3a$.
12. $3ab \div -3a$.
13. $20xy \div 5x$.
14. $-20xy \div 4y$.
15. $9ac \div -3ac$.
16. $6x \div -6x$.
17. $21y \div 21y$.
18. $24axy \div 12y$.

Divison is also expressed in fractional form, thus $\frac{4}{2}$ is read $4 \div 2$ or 4 over 2.

19. $\dfrac{6}{2}$.
20. $\dfrac{-6}{2}$.
21. $\dfrac{-10a^3}{-5}$.
22. $\dfrac{10a^3}{-5a}$.
23. $\dfrac{21x}{7}$.
24. $\dfrac{-21x^3}{7x}$.
25. $\dfrac{-16y^3}{4y^2}$.
26. $\dfrac{-18m^4}{6m^2}$.
27. $\dfrac{18n^3}{6n^3}$.
28. $\dfrac{-18s^5}{-6s}$.
29. $\dfrac{20r^3}{5r^2}$.
30. $\dfrac{-20r}{-5r^2}$.
31. $\dfrac{21b^7}{7b^3}$.
32. $\dfrac{12y}{3}$.
33. $\dfrac{-12c^5}{3c^5}$.
34. $\dfrac{9x^n}{-3x}$. *Ans.* $-3x^{n-1}$
35. $\dfrac{-48x^n}{-16x^2}$.
36. $\dfrac{-48x^n}{16x^m}$. *Ans.* $-3x^{n-m}$
37. $\dfrac{-20a^2}{4a}$.
38. $\dfrac{9a^{10}}{-3a^7}$.
39. $\dfrac{-14x^6}{-2x^2}$.
40. $\dfrac{4x^3}{x^2}$.
41. $\dfrac{16r^3}{-2r}$.
42. $\dfrac{10a^5b}{2a}$.
43. $\dfrac{-21a^2x^3}{7x^2}$.
44. $\dfrac{-24x^3y^2}{6xy^2}$.
45. $\dfrac{18m^4n^3}{-6mn^2}$.
46. $\dfrac{75x^2y^3}{-15x^2y}$.
47. $\dfrac{-85a^2c^5}{17a^2c^4}$.
48. $\dfrac{-98r^3s^4}{-7r^2s^4}$.
49. $\dfrac{105x^2y^5}{5xy^5}$.
50. $\dfrac{102m^2n}{6m^2n}$.
51. $\dfrac{78ab^2c^3}{-39abc^3}$.
52. $\dfrac{32r^3s}{16r^3}$.
53. $\dfrac{42m^2n^3}{-7n^3}$.
54. $\dfrac{-84x^2yz^3}{12x^2z^3}$.

Using the following couplets, first add; second subtract the lower from the upper number; third multiply; fourth divide the upper by the lower number. (Perform each operation on the first couplet and then on the second and so on.)

55. $\begin{array}{r}-12\\4\\\hline\end{array}$ 56. $\begin{array}{r}12\\-4\\\hline\end{array}$ 57. $\begin{array}{r}18\\9\\\hline\end{array}$ 58. $\begin{array}{r}-18\\-6\\\hline\end{array}$ 59. $\begin{array}{r}20\\-4\\\hline\end{array}$ 60. $\begin{array}{r}-20\\5\\\hline\end{array}$

61. $\begin{array}{r}-28a\\4a\\\hline\end{array}$ 62. $\begin{array}{r}-28b\\-4b\\\hline\end{array}$ 63. $\begin{array}{r}28c\\-4c\\\hline\end{array}$ 64. $\begin{array}{r}96x\\-16x\\\hline\end{array}$ 65. $\begin{array}{r}-84u\\-21u\\\hline\end{array}$ 66. $\begin{array}{r}-84s\\21s\\\hline\end{array}$

67. $\begin{array}{r}105k\\-15k\\\hline\end{array}$ 68. $\begin{array}{r}-91t\\7t\\\hline\end{array}$ 69. $\begin{array}{r}98x\\-7x\\\hline\end{array}$ 70. $\begin{array}{r}-63ab\\-21ab\\\hline\end{array}$ 71. $\begin{array}{r}-56m\\8m\\\hline\end{array}$ 72. $\begin{array}{r}72r\\12r\\\hline\end{array}$

73. $\begin{array}{r}15pq\\-10pq\\\hline\end{array}$ 74. $\begin{array}{r}-39y\\-13y\\\hline\end{array}$ 75. $\begin{array}{r}25rs\\-6rs\\\hline\end{array}$

44. **Division of a Polynomial by a Monomial.**—The division of a polynomial by a monomial corresponds to short division in arithmetic.

Illustration. $\begin{array}{l}2a^2)\underline{4a^3 - 6a^2b + 10a^3b^2}\\ \qquad 2a - 3b + 5ab^2\end{array}$ or expressed as a fraction $\dfrac{4a^3 - 6a^2b + 10a^3b^2}{2a^2} = 2a - 3b + 5ab^2$. It is seen that each term of the polynomial is divided by the monomial.

PROBLEMS

Solve the following: Check as in §31.

1. $(2a^2 + ab) \div a$.
2. $(-6a^3b + 12ab) \div 6ab$.
3. $(-4x + 6y - 2z) \div -2$.
4. $(4x^5 + 12x^2y) \div -4x^2$.
5. $\dfrac{18a^2b^5 + 16b^5}{-2b^5}$. $-9a^2$
6. $\dfrac{x^{10} - x^8 - x^6}{-x^5}$.
7. $\dfrac{x^{12} - x^{11} - x^9 + x^8}{-x^8}$.
8. $\dfrac{-16a^2b + 18ab^2 + 14a^2b^2}{-2ab}$.
9. $\dfrac{12m^2n^4 - 18m^5n^2 - 24m^3n^3}{-6m^2n^2}$.
10. $\dfrac{27y^9 - 9y^6 + 18y^5}{-9y^5}$.
11. $\dfrac{32abc + 16a^2b^2c^2}{-2abc}$.
12. $\dfrac{105x^2 - 45x^3 + 30x^4}{-15x}$.
13. $\dfrac{196r^3s^2 - 56r^2s^3}{14rs^2}$.
14. $\dfrac{-x - y - s - 1}{-1}$.

45. **Division of a Polynomial by a Polynomial.**—This process corresponds to long division in arithmetic.

Illustration.—1. Divide $-4x^3 - 25x^2 + x^4 + 48x - 12$ by $x^2 + 3x - 6$. The process will be learned best by following the work step by step.

$$\begin{array}{rrrrr|l}
\multicolumn{5}{c|}{\text{Dividend}} & \\
x^4 & -4x^3 & -25x^2 & +48x & -12 & x^2 + 3x - 6 \text{ Divisor} \\ \cline{6-6}
x^4 & +3x^3 & -6x^2 & & & x^2 - 7x + 2 \text{ Quotient.} \\ \cline{1-3}
 & -7x^3 & -19x^2 & +48x & & \\
 & -7x^3 & -21x^2 & +42x & & \\ \cline{2-4}
 & & 2x^2 & +6x & -12 & \\
 & & 2x^2 & +6x & -12 & \\ \cline{3-5}
\end{array}$$

Explanation step by step.

1. The dividend and the divisor are arranged according to the descending powers of x.

2. Divide the first term of the divisor, $\mathbf{x}^2$, into the first term of the dividend, $\mathbf{x}^4$, to obtain the first term of the quotient, $\mathbf{x}^2$.

3. Multiply each term of the divisor by the first term of the quotient and place the product under the dividend. Subtract and bring down the next term of the dividend.

4. Divide the first term of the divisor, $\mathbf{x}^2$, into the first term of the remainder, $-7\mathbf{x}^3$, and obtain the second term of the quotient, $-7\mathbf{x}$.

5. Multiply each term of the divisor by $-7\mathbf{x}$ and place the product as before.

6. Continue until there is no remainder or until the first term of the divisor is no longer contained in the first term of the remainder last obtained.

7. Check.—The divisor times the quotient + the remainder, if any, = the dividend. When the work is arranged as shown the multiplication is easily performed without rewriting the divisor and the quotient. Long division and the check by multiplication give an excellent review in the four fundamental operations of signed numbers.

Check the above problem by substituting $x = 2$.

Thus,

$$\begin{array}{r|l} \text{Dividend} & \\ -32 & 4 \text{ Divisor} \\ \cline{2-2} & -8 \text{ Quotient.} \end{array}$$

Note.—*Do not substitute a numerical value which will result in division by zero.*

Thus $6 \div 0$ does not give any known number, for any number multiplied by zero equals zero.

Also since $6 \times 0 = 0$ and $15 \times 0 = 0$, $0 \div 0$ may equal 6, 15 or in fact any number.

Illustration.—2. Compare the process of division of arithmetical numbers with that of literal numbers.

(*a*) Dividend

$$\begin{array}{r|l} 227488 & 921 \text{ Divisor} \\ \cline{2-2} 1842 & 247 \text{ Quotient} \\ \cline{1-1} 4328 & \\ 3684 & \\ \cline{1-1} 6448 & \\ 6447 & \\ \cline{1-1} 1 \text{ Remainder.} & \end{array}$$

(*b*) Dividend

$$\begin{array}{r|l} 6a^2 + ab - 3b^2 & 2a - b \text{ Divisor} \\ \cline{2-2} 6a^2 - 3ab \quad\quad & 3a + 2b \text{ Quotient} \\ \cline{1-1} 4ab - 3b^2 & \\ 4ab - 2b^2 & \\ \cline{1-1} - b^2 \text{ Remainder.} & \end{array}$$

Check

(a)

$$\begin{array}{rl}
921 & \text{Divisor} \\
247 & \text{Quotient} \\
\hline
6447 & \\
3684 & \\
1842 & \\
\hline
227487 & \\
1 & \text{Remainder} \\
\hline
227488 & \text{Dividend.}
\end{array}$$

(b)

$$\begin{array}{l}
2a - b \text{ Divisor} \\
3a + 2b \text{ Quotient} \\
\hline
6a^2 - 3ab \\
\qquad + 4ab - 2b^2 \\
\hline
6a^2 + ab - 2b^2 \\
\qquad\qquad - b^2 \text{ Remainder} \\
\hline
6a^2 + ab - 3b^2 \text{ Dividend.}
\end{array}$$

PROBLEMS

Divide. Check the results by §31 or by the above method.

1. $a^2 + 5a + 6$ by $a + 2$.
2. $a^2 + 7a + 12$ by $a + 3$.
3. $a^2 - 5a + 6$ by $a - 2$.
4. $x^2 - 4x + 4$ by $x - 2$.
5. $2x^2 - 5x - 12$ by $x - 4$.
6. $a^2 - 2ab + b^2$ by $a - b$.
7. $a^2 + 2ab + b^2$ by $a + b$.
8. $x^2 + x - 56$ by $x - 7$.
9. $6a^2 + ab - b^2$ by $3a - b$.
10. $4x^2 - 9$ by $2x - 3$.
11. $x^2 - 4y^2$ by $x + 2y$.
12. $x^4 - y^4$ by $x^2 + y^2$.
13. $x^2 + 2x - 63$ by $x + 9$.
14. $4x^2 + 2x - 30$ by $2x - 5$.
15. $3x^2 - 5x + 2$ by $3x - 2$.
16. $1 - 4x^2$ by $1 - 2x$.
17. $1 - a - 2a^2$ by $1 + a$.
18. $12 + 5y - 3y^2$ by $4 + 3y$.
19. $d^3 - 1$ by $d - 1$.
20. $d^3 + 1$ by $d + 1$.
21. $x^3 - y^3$ by $x - y$.
22. $x^3 + y^3$ by $x + y$.
23. $a^6 - b^6$ by $a^3 - b^3$.
24. $x^3 + x^2 - x - 1$ by $x - 1$.
25. $2m^3 - m^2n + 11mn^2 + n^3$ by $2m + n$.
26. $12a^2 - 29ab - 14b^2$ by $4a - 7b$.
27. $54x^2 - 15xy - y^2$ by $3x - y$.
28. $96r^2 - 52rs - 63s^2$ by $8r + 9s$.
29. $w^3 - 7w^2 + 10w$ by $w - 2$.
30. $65k^2 + 181kl + 126l^2$ by $5k + 7l$.
31. $18a^2y^2 + 36ay + 10$ by $3ay + 5$.
32. $2a^4b^2 + 7a^2b - 18$ by $2a^2b + 9$.
33. $6a^2m^4 - 23am^2 + 20$ by $3am^2 - 4$.
34. $a^4b^4 + 2a^2b^2c + c^2$ by $a^2b^2 + c$.
35. $1 + 2mn^2 + m^2n^4$ by $1 + mn^2$.
36. $2a^3 + 3a^2 + 3a + 1$ by $2a + 1$.
37. $8x^3 + 27$ by $2x + 3$.
38. $x^4 + x^2 + 1$ by $x^2 + x + 1$.
39. $r^3 + 3r^2s + 3rs^2 + s^3$ by $r + s$.
40. $x^4 + y^4$ by $x + y$.
41. $m^5 + n^5$ by $m + n$.
42. $a^8 - 1$ by $a^2 + 1$.
43. $2x^6 + 3x^5y - 2x^4y^2 - 9x^3y^3 - 3x^2y^4 + 6xy^5 - 2y^6$ by $x^2 - 2y^2$.
44. $a^9 - 1$ by $a^6 - a^3 + 1$.
45. $x^{13} + 1$ by $x + 1$.
46. $6x^6 - x^4 + 20x^3 - 2x^2 + 8x + 6$ by $2x^3 - x + 6$.
47. $a^8 - y^8$ by $a^2 + y^2$.
48. $9a^3 - 7a + 8a^4 + 12a^2 + 6 + 6a^5$ by $-2a + 3a^2 + 1$. (Re-arrange terms.)

49. $6m^5 + 2m^4 + m^2n^4 + 6m^3n + 2m^2n + n^5$ by $m^2 + n$.

50. $\frac{1}{1 - a} = ?$ to five terms and a remainder.

46. Meaning of the Zero Exponent.—Consider the division, $\frac{a^2}{a^2} = a^{2-2} = a^0$. Again, $\frac{a^2}{a^2} = 1$, for any expression divided by itself equals 1.

Therefore, if $\frac{a^2}{a^2} = a^0$ and $\frac{a^2}{a^2} = 1$, it follows that $a^0 = 1$.

In general $\frac{x^n}{x^n} = x^0 = 1$.

There is evidently only one value to give to an expression with a zero exponent, and that is 1.

PROBLEMS

Find the value of the following:

1. b^0. 2. 5^0. 3. $(\frac{1}{3})^0$. 4. k^0. 5. z^0. 6. $\frac{b^2}{b^0}$.

7. $\frac{a^0}{x^0}$. 8. $\frac{r^4m^2}{x^0}$. 9. $\frac{2^0x^2y^0}{x}$. 10. $\frac{3a^0b}{c^0}$. 11. $\frac{5(a - b)^0}{3}$. 12. $\frac{6^0(x + y)^0}{5^0}$.

47. Negative Exponents.—By division $\frac{a^5}{a^2} = a^{5-2} = a^3$.

Here we subtract the exponent 2 of the divisor, or denominator, from the exponent 5 of the dividend, or numerator, and get the exponent 3 for the quotient.

By division $\frac{a^2}{a^5} = a^{2-5} = a^{-3}$. Again we use the same rule, but note that $2 - 5 = -3$, and therefore the exponent of the quotient this time is negative. Study the following:

1. $\frac{a^5}{a^7} = a^{-2}$. 2. $\frac{x^4}{x^5} = x^{-1}$. 3. $\frac{a^3}{a^6} = a^{-3}$. 4. $\frac{y^{10}}{y^{12}} = y^{-2}$.

PROBLEMS

Find the value of the following:

1. $\frac{m^2}{m^3}$. 2. $\frac{r^2}{r^3}$. 3. $\frac{b^5}{b^{10}}$. 4. $\frac{c^{12}}{c^{13}}$. 5. $\frac{e^5}{e^{12}}$.

6. $\frac{x^3}{x^5}$. 7. $\frac{y^{10}}{y^{11}}$. 8. $\frac{a^8}{a^{-1}}$. 9. $\frac{n^{-5}}{n^{-6}}$. 10. $\frac{s^{-2}}{s^{-3}}$.

11. $\frac{t}{t^{-1}}$. 12. $\frac{w^{-8}}{w^{-6}}$. 13. $\frac{x^a}{x^b}$. 14. $\frac{y^m}{y^n}$. y^{m-n} 15. $\frac{x^{a+b}}{x^{a-b}}$.

Zero and negative exponents will be treated more fully in a later chapter.

CHAPTER III

SIMPLE EQUATIONS

48. **Equations.**—An equation is the statement of equality between two expressions.

Illustrations.—1. If we say that the weight of one bolt added to the weight of another bolt equals the weight of a third bolt, we have made a statement of equality. If we represent these weights by symbols as $\mathbf{w}_1 = wt$ of the first bolt, $\mathbf{w}_2 = wt$ of the second bolt and $\mathbf{w}_3 = wt$ of the third bolt, we have $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_3$.

2. We know that the distance one can walk is equal to the number of hours walked multiplied by the number of miles per hour. Thus if $\mathbf{d} =$ distance, $\mathbf{t} =$ number of hours, and $\mathbf{r} =$ the rate per hour, we have the equation $\mathbf{d} = \mathbf{rt}$.

3. The time required to go a certain distance is the distance divided by the rate of travel. Thus $\mathbf{t} = \dfrac{\mathbf{d}}{\mathbf{r}}$.

4. Again, the rate of speed is found by dividing the distance by the time, thus $\mathbf{r} = \dfrac{\mathbf{d}}{\mathbf{t}}$.

The expression to the left of the equality sign is the **first member** of the equation and the expression to the right of the equality sign is the **second member.**

49. **Quantities.**—A known quantity is an expression whose value is given or known. An unknown quantity is one whose value is to be found. Thus in $3x = 12$, 3 and 12 are known, x is unknown.

50. **Identity.**—When an equality is true for any value of the unknown quantity involved it is an identity. As $2t + 3t = 5t$. This reduces to $5t \equiv 5t$. Test this for several values of t.

51. **Conditional Equation.**—When the expressed equality is true only for a certain value of the unknown quantity involved it is a conditional equation.

Illustration. $3v + 1 = 2v + 4$ (v is the unknown quantity). This equation is true when $v = 3$ and for no other value of v. Substituting 3 for v, we have $3(3) + 1 = 2(3) + 4$ or $9 + 1 = 6 + 4$, *i.e.*, $10 \equiv 10$.

This last operation is testing or checking the value of v in the given equation. The substitution must always be made in the original equation.

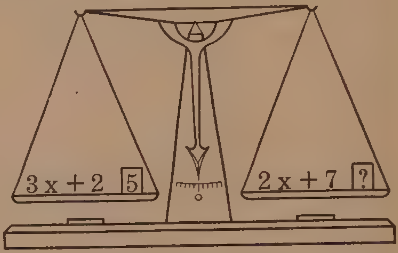
Important Note.—*The student must learn to check his work very early in the course. He must avoid the weakness of having to consult an answer book or of depending upon the teacher or fellow student to determine whether or not the result is correct. He should make a practice of checking his work and thus gain confidence in his ability to solve problems.*

Every conditional equation reduces to an identity when the value on which the condition depends is substituted for the unknown quantity.

52. **Solution.**—Solving an equation is the process of finding the value of the unknown quantity which will satisfy the given condition, *i.e.*, the value which, when substituted for the unknown quantity, reduces the equation to an identity.

53. **Root.**—The value of the unknown quantity obtained by the solution is the root of the equation.

54. **Use of the Equation.**—One of the objects of algebra is to simplify the solution of problems. The student must learn to translate given conditions and statements into simple algebraic equations and then find a solution of these equations. In other words the student must form an equation between the known and unknown quantities and solve for the unknown quantity.



The meaning of the equation will be readily understood if we think of the two members of the equation as being balanced on the equality sign as a support. Keep in mind that as long as the balance is not disturbed the equation remains true. Thus, if $3x + 2 = 2x + 7$ and 5 is added to the first member we must add 5 to the second member. The same principle is true in case of subtraction, multiplication, and division. Hence, if one member of an equation is changed, the other member of the equation must be changed by the same operation or the equality will be destroyed.

55. **Axioms.**—An axiom is a self evident truth. The following axioms will be frequently used in solving equations, as well as in studying relations between geometric figures. Let the student cite numerous examples to illustrate the following axioms.

1. If equals be added to equals the sums are equal. Thus $7 = 7$. If 2 be added to each member we have $9 = 9$.
2. If equals be subtracted from equals the remainders are equal.
3. If equals are multiplied by equals the products are equal.
4. If equals are divided by equals the quotients are equal. (The divisor must not $= 0$.)
5. A quantity may be substituted for its equal in any process. Thus, if $a = b$ and $3a + 2 = 5a$ we may write $3b + 2 = 5b$.
6. Things equal to the same thing are equal to each other.
7. Things equal to equal things are equal to each other.
8. If equals are added to unequals the sums are unequal in the same order.
9. If equals are subtracted from unequals the remainders are unequal in the same order.

10. If unequals are subtracted from equals the remainders are unequal in the reverse order.

11. If unequals are multiplied or divided by equals, the results are unequal in the same order.

12. The whole is equal to the sum of its parts.

13. The whole is greater than any of its parts.

56. **Application of Axioms in Solving Equations.**

Illustrations.

1. Solve $3r = 15$.

Solution. $3r = 15$.

Divide both members by 3. *Ax.* 4.

Then $r = 5$.

Check

$3(5) = 15$

$15 \equiv 15$

2. Solve $8t = 5t + 6$.

Solution. $8t = 5t + 6$.

Subtract $5t$ from both members. *Ax.* 2.

Then $8t - 5t = 6$, or $3t = 6$.

$\therefore t = 2$. *Ax.* 4.

Check by substituting in the original equation.

3. Solve $5x - 4 = 3x + 2$.

Solution. $5x - 4 = 3x + 2$.

Subtract $3x - 4$ from both members. *Ax.* 2.

Then $5x - 3x = 2 + 4$.

Collect terms. $2x = 6$.

$\therefore x = 3$. *Ax.* 4.

4. Solve for y in $y - b = mx$.

Solution. $y - b = mx$.

Add b to both members. *Ax.* 1.

Then $y = mx + b$.

5. Solve $3q + 8 = 2(5q - 2)$.

Solution.—$3q + 8 = 2(5q - 2)$.

Remove parentheses. $3q + 8 = 10q - 4$.

Subtract $3q - 4$ from both members. *Ax.* 2

Then $12 = 7q$.

$\therefore \frac{12}{7} = q$. *Ax.* 4., *i.e.*, $q = \frac{12}{7}$.

57. **Transposition.**—Transposition is the process of removing terms from one member of an equation to the other member. By studying illustrations 2 to 5 inclusive we see that the effect of using axioms 1 and 2 was to bring to one member of the equation all terms containing the unknown quantity and to the other member all known terms. In the transposition, however, every term taken from one side of the equality sign to the other had the sign changed. We may, therefore, adopt the general principle. A term or quantity may be transposed from one member of an equation to the other provided the sign is changed.

58. **Change of Signs.**—The sign of every term in both members of an equation may be changed without effecting the equation, *i.e.*, both members of the equation may be multiplied by -1. *Ax.* 3.

59. **General Directions for Solving Simple Equations.**

1. *Remove parentheses, if any occur.*
2. *Transpose all terms containing the unknown number to one member of the equation, and all known terms to the other member.*
3. *Collect like terms.*
4. *Divide both members by the coefficient of the unknown number.*
5. *Check your result by substituting the root in the original equation.*

PROBLEMS

Solve and check the following:

1. $7t = 56$.
2. $4r = -12$.
3. $1.5x = 30$.
4. $7m = 38$.
5. $.25s = 6.25$.
6. $.5p = 1$.
7. $-3x = -15$.
8. $13x = 0$.
9. $15n - 8 = 8n + 20$.
10. $4f + 56 = 17f - 100$.
11. $3y - 13 = 14y + 12$.
12. $\frac{3}{4}x = 15 - \frac{1}{4}x$.
13. $\frac{3}{2}y = 5 - \frac{1}{2}y$.
14. $2(x + 7) = 27$.
15. $2x - (3x + 4) = 3(x + 2) - 5$.
16. $5(s + 12) - 7s = 40$.
17. $V - 16 = 32$.
18. $\frac{3}{2}x = 12$.
19. $\frac{4}{5}n = 16$.
20. $\frac{5}{8}V = 25$.
21. $90 = \frac{1}{2}V + 4$.
22. $180 = 5s$.
23. $1032 = 13.2 + 18f$.
24. $360 = 45 + 16t$.
25. $450 = 37 + 32t$.
26. $372 - 160 = 15a$.

Evaluate the following formulas. Check.

27. $C = 25 + 2(n - 10)$. If $n = 17$, find C.
If $C = 55$, find n.

28. $C = \frac{5}{9}(F - 32)$. If $F = 77$, find C.

29. $s = \frac{1}{2}gt^2$. If $g = 32$ and $t = 3$, find s.
If $g = 32$ and $s = 1024$, find t.

30. $s = vt$. If $v = 8640$ and $t = 3$, find s.
If $v = 186$ and $s = 837$, find t.

31. $s = vt + \frac{1}{2}gt^2$. If $g = 32$, $t = 5$, $v = 196$, find s.

32. $s = \frac{1}{2}a(2t - 1)$. If $a = 24$, $t = 8$, find s.
If $s = 300$, $a = 12$, find t.

33. $F = \dfrac{Mv^2}{r}$. If $M = 75$, $v = 15$, $r = 25$, find F.
If $F = 725$, $v = 5$, $r = 20$, find M.

34. $F = M{\cdot}a$. If $M = 120.5$, $a = 8.5$, find F.
If $F = 733.6$, $a = 11.2$, find M.

35. $PV = C$. If $C = 2890.89$, $V = 34.83$, find P.
If $C = 3177$, $P = 105.9$, find V.

36. $V = lwh$. If $l = 6.2$, $w = 5.3$, $h = 3.9$, find V.
If $V = 335$, $l = 12.5$, $w = 6.7$, find h.
If $V = 774.04$, $l = 15.2$, $h = 8.9$, find w.

37. $A = \frac{1}{2}h(b_1 + b_2)$. If $h = 6$, $b_1 = 13.2$, $b_2 = 10.9$, find A.
If $A = 96$, $h = 10$, $b_1 = 11.3$, find b_2.
If $A = 139.2$, $h = 12$, $b_2 = 9.3$, find b_1.

38. $y = mx + b$. If $m = 2$, $b = -3$, find y.
If $m = -5$, $b = 4$, find y.

60. **Algebraic Solution of Problems.**—Many problems occur which are difficult and indeed impossible to solve from the knowledge of arithmetic. We have now learned that letters may be used to represent numbers and that we can perform all the operations of arithmetic using letters in place of numbers. We may, therefore, use a letter for an unknown value and by means of the equation find the value which at first was unknown.

A few simple relations will aid a great deal in the algebraic statement of the problem to be solved. Very often when general relations are given the student can think of a similar example in arithmetic.

Illustrations.—1. At 20 miles per hour, how many hours will it take to travel 60 miles? *Ans.* $60 \div 20$ or 3 hours.

2. At 2 miles per hour, how many hours will it take to go a distance of s miles? *Ans.* $\frac{s}{2}$ hrs.

ORAL PROBLEMS

1. If n represents an odd integer (whole number), what kind of a number is $n + 1$? $n + 2$? $n - 1$? $n - 2$? $2n$? $2n - 1$? $2n + 1$?

2. Beginning with n as an integer, name three consecutive (numbers which follow in order as 5, 6, and 7) numbers; three consecutive odd numbers; three consecutive even numbers.

3. Let n represent the sum of two numbers. If one of them is 5, what is the other?

4. By how much does 20 exceed 10?

5. By how much does a exceed b?

6. What part of 30 is 6? Of a is b?

7. If a man is 40 years old, how old will he be a years hence? How old was he b years ago?

8. How far can a person travel in t hours at v miles per hour?

9. Express the number 72 in units; the number whose units' digit is u and whose tens' digit is t in units. Express each of the above numbers with the digits in the reverse order.

10. If x represents 10, what number is represented by $4x + 3$? By $3x^2 + 2x + 1$?

11. Express a dollars in cents; b dimes in cents; c quarters in cents. Express the sum of a dollars, c quarters, and b dimes in cents.

12. Express a feet plus b inches in inches.

13. Express x degrees plus y minutes in minutes.

14. Give an expression for the perimeter of a rectangle, if it is twice as long as it is wide. Draw a figure.

15. If x represents a number, express 5 less than 3 times the number.

16. The sum of two angles is 90°, one of them is represented by a degrees. How many degrees in the other?

61. **Problems Which Lead to Simple Equations.**

Illustrations.—1. Find three consecutive numbers whose sum is 39.

Solution.—Let $x =$ the first number. (1)
Then $x + 1 =$ the second number. (2)
And $x + 2 =$ the third number. (3)

Adding (1), (2), and (3),

$3x + 3 =$ the sum of the three numbers. (4)

$\therefore 3x + 3 = 39.$ (5)

$3x = 39 - 3.$ Why?

$3x = 36.$ (6)

$x = 12$, the first number. (7)

$x + 1 = 13$, the second number. (8)

$x + 2 = 14$, the third number. (9)

Check. $12 + 13 + 14 = 39.$

2. A father is three times as old as his son. Ten years hence he will be twice as old as his son. Find the age of each.

Solution. Let $x =$ the son's age now. (1)

Then $x + 10 =$ the son's age 10 years hence. (2)

And $2(x + 10) =$ twice the son's age 10 years hence. (3)

$3x =$ the father's age now. (4)

$3x + 10 =$ the father's age 10 years hence. (5)

$\therefore 3x + 10 = 2(x + 10).$ (3) and (5)

$3x + 10 = 2x + 20.$ Why?

$3x - 2x = 20 - 10.$ Why?

$x = 10$, the son's age.

$3x = 30$, the father's age.

Check.—First, 30 is 3 times 10. Second, 10 years hence the father will be 40 years old and the son will be 20 years old. 40 is twice 20.

Note.—*Check these problems by the conditions given in the problem and not by substituting in the equations formed from those conditions. You may have stated some condition incorrectly.*

3. A certain number has two digits. The digit in the tens' place is 2 less than twice the digit in the units' place. If 27 is subtracted from the number the result is a number with the digits reversed.

Solution. Let $x =$ units' digit. (1)

Then $2x - 2 =$ the tens' digit. (2)

And $10(2x - 2) + x =$ the value of the number. (3)

Also $10x + (2x - 2) =$ the value of the number with the digits reversed. (4)

$\therefore 10(2x - 2) + x - 27 = 10x + (2x - 2).$ (5)

Solving this equation, $x = 5.$

And $2x - 2 = 8.$

Hence the number is 85.

Check. 8 is 2 less than twice 5 and $85 - 27 = 58.$

PROBLEMS

1. Find three consecutive numbers whose sum is 54.
2. Find three consecutive odd numbers whose sum is 81.
3. Find three consecutive even numbers whose sum is 156.

4. A rectangle is three times as long as it is wide. The perimeter is 200 yards. Find the dimensions. *Ans.* 25 yds. and 75 yds.

5. A rectangle is 50 feet long. If it were 4 feet wider it would contain 1700 square feet. Find the width.

6. Divide 42 into two parts, so that one part shall be twice the other.

7. A tank which is $\frac{5}{8}$ full contains 120 gallons. Find the capacity.

8. If 80 is added to a certain number it becomes five times as great as originally. Find the number.

9. The width of the Horseshoe Fall at Niagara is 160 feet less than 20 times the height. The width and height together are 3158 ft. Find the width and the height. *Ans.* 3000 ft. and 158 feet.

10. Eiffel Tower, Paris is 430 feet higher than the Washington Monument, Washington, D. C. Their combined height is 1540 ft. Find the height of each.

11. One-third of a tower is constructed of stone, one-half of brick and the remainder 30 feet of steel. How high is the tower?

12. A telegram at the rate of 40 cents for the first 10 words, and 3 cents for each additional word cost 61 cents. How many words did it contain?

13. At the same rate as in Prob. 12, how many words can be sent for 76 cents?

14. The total length of the streets in Salt Lake City is 478 miles. The length of the streets in miles is 50 less than 4 times the standard width of a street in feet. Find the width of a street in feet. *Ans.* 132 ft.

15. The elevation of a Bench Mark on a step of Memorial Hall in Fairmount Park is 9.365 feet more than the Bench Mark at 41st St. and Parkside Ave., Phila. A Bench Mark at Horticultural Hall is 3.501 feet less than the one at Memorial Hall. The sum of the three elevations is 328.762 feet. Find the elevation of each.

16. A merchant has $9.50 in dimes and quarters. How many coins of each denomination has he if there are 50 pieces?

17. Divide a line 45 feet long into three parts, so that the second shall be 3 feet longer than the first, and the third one and one-half times as long as the first.

18. A man walked a certain distance, rode 10 miles by automobile, and went by train five times as far as he had previously gone. He traveled 85 miles. How far did he walk? How far did he go by train?

19. A factory floor is 30 feet longer than it is wide. The perimeter is 540 feet. Find the dimensions. Find the area.

20. Two lots are of equal area. The width of the first is 55 feet and that of the second 50 feet. The second is 10 feet longer than the first. Find the length of each lot. Check by finding their areas.

21. Find the velocity of a body after falling 5 seconds. Hint $v = gt$, *let* $g = 32$.

22. The distance a freely falling body falls in any one second is given by $s = \frac{1}{2}g(2t - 1)$. During what second will a body fall 528 feet?

23. How far will a freely falling body fall in the 5th second?

24. The velocity of sound in air at the freezing temperature (0°C. or 32°F.) is about 1,090 feet per second. The velocity increases 2 feet per second for each degree C of rise in temperature. Write a general formula for the velocity of sound in air using C. Find the velocity of sound in air at 10°C., 15°C., 20°C., 23°C., 25°C. *Ans.* $V = 1{,}090 + 2C$.

25. The velocity of sound in air increases 1.12 feet per second, for each degree F. of rise in temperature. Write the general formula for velocity of sound in air and find the velocity of sound in air at 60°F., 75°F., 90°F., 98°F., 100°F. *Ans.* $V = 1{,}090 + 1.12(F - 32°)$.

26. At 8°C. sound travels 10,142 feet per second faster in glass than in water. The velocity in glass is 726 feet more than three times as much as in water; the velocity in iron is 2,012 feet per sec. less than four times as much as in water. Find the velocity of sound in water; in glass; in iron at 8°C.

27. The cost of sewerage for a certain property with a frontage of 100 feet was $298.85. The distance from the house to the property line is 60 feet and from the property line to the main sewer 37.5 feet. The cost of connection from the property line to the main sewer was 25 cents a foot more than from the house to the property line. The cost per front foot for the main sewer was 30 cents a foot more than twice the cost from the house to the property line. The riser and trap cost $6.60. Find the cost, per foot, of each section. Draw a diagram.

28. Heavy concrete for basements is made of the following materials in bulk, 1 part cement, 3 parts sand and 6 parts of stone or gravel. 122 cu. ft. of material were used. How many cu. ft. of each was used? (There will not be 122 cu. ft. of concrete.)

29. Cinder concrete for cellar floors is made of 1 part cement, $2\frac{1}{2}$ parts sand, and 5 parts of cinder. If 170 cu. ft. of material was used, find the number of cu. ft. in cement, sand and cinders.

30. If the concrete in Prob. 29 is made of 1 part cement, 3 parts of sand and 6 parts of cinder, find the number of cu. ft. of each material used.

62. **Further Use of Equations.**—The sequence of topics in this text is somewhat out of the usual order. Thus far the equation has been applied to the solution of the usual algebraic problems. It is, however, of great advantage and indeed a necessity for solving many problems in geometry.

Therefore it is necessary at this point to define Geometry and some geometric terms.

63. **Geometry.**—The word geometry is derived from two Greek words meaning earth, and to measure. The science was first used for measuring land, hence its name. Geometry is that branch of mathematics which deals with the investigation of the relations, properties, and measurements of points, lines, angles, surfaces, and solids.

64. **Point.**—A point has position only. It is indicated by a dot as $\cdot A$. Capital letters are used to denote position. Small letters are used to denote magnitude.

65. **Lines.**—The path of a moving point is a line. When a point in motion does not change its direction it traces a **straight line.** A line means a straight line unless otherwise indicated. A **broken line** is made up of a series of connected straight lines. When a point in motion changes its direction at every instance it traces a **curved line.** A line may be indefinite in length or it may be limited at any point. A line is designated by two letters usually at its end points as . This is a line of definite length. A line is often designated by a single letter as .

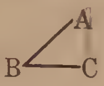
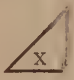
Test the edge of a ruler by drawing a line along its edge. Turn the ruler over on your paper and see if the edge coincides with the line drawn. If it does, the edge is straight. Make the same test with a piece of card board whose edge is not straight. What is the result?

66. **Surfaces.**—When a line moves, except in its own direction, it generates a surface. A **plane surface,** or a **plane,** is a surface such that when a straight edge is applied in any position every part of it will touch the surface. A **curved surface** is a surface no part of which is plane.

When will a straight line generate a plane surface? A curved surface? How would you test a plane surface with a straight edge?

67. **Solid.**—A surface which moves, except in its own direction, generates a solid. Give illustrations.

68. **Summary.**—A point has no dimensions. A line has one dimension, length. A surface has two dimensions, length and breadth. A solid has three dimensions, length, breadth, and height.

69. **Angles.**—An angle is the amount of opening between two lines which meet at a point, as  or  and is read angle ABC or angle x. The sign ∠ means angle. BA and BC are the sides, and the point B is the vertex of the angle. The letter at the vertex is always placed between the other two when reading an angle.

An angle is generated by rotating a line about a fixed point in a plane. Imagine two lines coinciding and hinged at one end. Keep one line fixed and rotate the other about the fixed point in the direction opposite to the movement of the hands of a clock. Stop anywhere you choose and there is an opening between the lines as in the figures above. This is an angle. When generated counter-clockwise it is considered a positive angle, when generated clockwise, a negative angle. This can be illustrated by use of a pair of compasses.

70. **Bisector.**—The bisector of an angle is the line which divides the angle into two equal angles.

71. **Unit Angle.**—The unit angle is 1°. It is the one three-hundred-sixtieth part of the angular magnitude about a point. 60 sec. = 1 min. 60 min. = 1°.

72. **Straight Angle.**—When the opening is such that the two sides of an angle extend in opposite directions from the vertex, thus forming a straight line, the angle is a straight angle. It contains 180°. All straight angles are equal.

73. **Adjacent Angles.**—When two angles have the same vertex and a common side between them they are adjacent angles.

74. **Right Angle.**—When one straight line meets another straight line making the two adjacent angles equal, they are right angles. Therefore, a right angle is half a straight angle and equals 90°. All right angles are equal. *Ax.* 4.

75. **Perpendiculars.**—When two lines meet at right angles they are perpendicular to each other, *i.e.*, **mutually perpendicular.** Therefore, at a given point on a straight line only one perpendicular can be drawn to the line.

76. **Oblique Angles.**—All angles that are not right angles or straight angles are oblique angles.

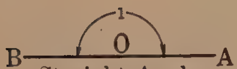
77. **Acute Angle.**—An acute angle is an angle less than 90°.

78. **Obtuse Angle.**—An obtuse angle is an angle greater than 90° and less than 180°.

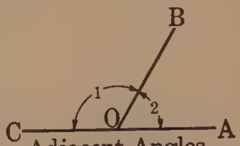
79. **Reflex Angle.**—A reflex angle is an angle greater than 180° less than 360°.

80. **Perigon.**—A perigon is an angle of 360°.

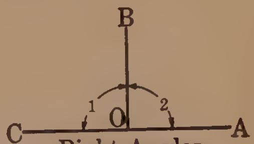
81. **Illustrations of Angles.**



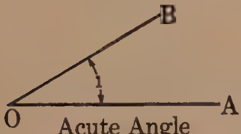
Straight Angle



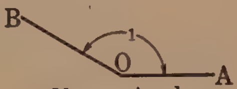
Adjacent Angles



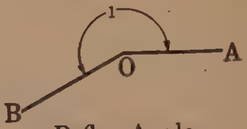
Right Angles



Acute Angle



Obtuse Angle



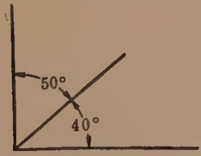
Reflex Angle

82. **Complementary Angles.**—Two angles whose sum is 90° are complementary. Thus, 40° is the complement of 50°, also 50° is the complement of 40°.

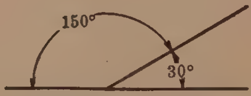
Give other illustrations.

Complements of the same angle or of equal angles are equal. *Ax.* 2.

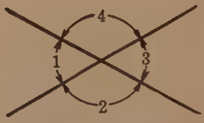
83. **Supplementary Angles.**—Two angles whose sum is 180° are supplementary. Thus, 30° is the supplement of 150°, also 150° is the supplement of 30°. Give other illustrations. Supplements of the same angle or of equal angles are equal. *Ax.* 2.



Complementary Angles



Supplementary Angles



Vertical Angles

84. **Vertical Angles.**—Two angles which have a common vertex, and the sides of one are the prolongations of the sides of the other are vertical angles. $\angle 1$ and $\angle 3$ are vertical angles, $\angle 2$ and $\angle 4$ are vertical angles.

Problem $\angle 1 + \angle 2 = 180°.$ §83.

$\angle 2 + \angle 3 = 180°.$

$\therefore \angle 1 + \angle 2 = \angle 2 + \angle 3.$ *Ax.* 6.

$\therefore \angle 1 = \angle 3.$ *Ax.* 2.

Show that $\angle 2 = \angle 4.$

We see from the above problem that when two straight lines intersect the vertical angles are equal.

85. **Theorem.**—A theorem is a statement which requires proof.

Illustration.—When two straight lines intersect the vertical angles are equal. This statement is naturally divided into two parts, the **hypothesis** and the **conclusion.** The hypothesis is that which is given as "when two straight lines intersect." The conclusion is that which is to be proved as "the vertical angles are equal."

PROBLEMS

Check where possible

1. Draw the following angles: Acute, obtuse, reflex, right, straight, oblique, adjacent, complementary, complementary-adjacent, complementary-nonadjacent, adjacent not complementary, supplementary, supplementary-adjacent, supplementary non-adjacent, adjacent not supplementary, vertical angles.

2. Draw two lines making the adjacent angles equal. What kind of angles are formed? What is the relation between these lines? How many perpendiculars can be drawn to a line at a given point? Why?

3. One angle is 75° 24′ 33″ and another is 50° 23′ 45″. Find their sum; their difference. Multiply each of them by 2. Divide each by 2.

4. How many degrees between the hands of a clock at 2 o'clock? At 4 o'clock? At 2:30 o'clock? At 8:45 o'clock? At 2:15 o'clock? At 12 o'clock?

5. Subtract 32° 25′ 35″ from 90°. The result is the complement of 32° 25′ 35″. Find the supplement of 32° 25′ 35″.

6. What is the complement of an angle of 30°? 42° 25′? 68° 32′ 21″? Find the supplement of each of these angles.

7. Write an equation stating that p and q are complementary angles. Solve the equation for p; for q.

8. Write an equation stating that r and s are supplementary angles. Solve for r; for s.

9. Find the angle which equals (check results):
(*a*) 4 times its complement.
(*b*) 10° less than its complement.
(*c*) 30° less than 5 times its complement.
(*d*) 9 times its supplement.
(*e*) 20° more than 3 times its supplement.
(*f*) 27° more than its supplement.
(*g*) $\frac{1}{5}$ of its complement; $\frac{1}{3}$ its supplement.

10. Find the angle, the sum of whose complement and supplement is 110°; 200°; 216°.

11. The sum of three angles is a perigon. The first is twice the second, the second is three times the third. How many degrees in each? Hint. *Let* x = *the value of the third angle.*

12. How many degrees in each of two complementary angles if the greater exceeds the less by 17°?

13. An angle is 30° less than its complement. What part of a right angle is it? What part of a straight angle? What part of its complement? What part of its supplement?

14. If the angle $x + 26°$ is the complement of $2x + 22°$, find the angles.

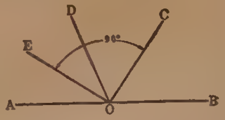
15. If the angles in Prob. 14 are supplementary, find them.

Note.—*Draw figures for problems* 16 *to* 21.

16. The angles about a point on one side of a straight line are represented as follows: Find them.
(*a*) $(2x + 27)°$, $(3x - 15)°$, and $(x + 42)°$.
(*b*) $(3x + 5)°$, $(6x - 3)°$, $(5x - 4)°$, and $(3x - 12)°$.
(*c*) $(2.7y + 36.43)°$, $(1.3y + 7.32)°$, and $(y - 32.25)°$.

17. The bisectors of two supplementary adjacent angles form a right angle.

Solution. $\angle AOD$ and $\angle DOB$ are supplementary.



$$\begin{aligned} \text{Then, } \angle AOD + \angle DOB &= 180^\circ \\ \text{Hence, } \tfrac{1}{2}\angle AOD + \tfrac{1}{2}\angle DOB &= 90^\circ \\ \angle EOC &= 90^\circ \qquad Ax.\ 4. \end{aligned}$$

18. Find the angle formed by the bisectors of two complementary adjacent angles.

19. If 50° and 60° are two adjacent angles, what angle is formed by their bisectors?

20. If **m** and **n** are any two adjacent angles, what angle is formed by their bisectors?

21. Find the number of degrees in each of the following pairs of vertical angles.

(*a*) $(5x + 15)^\circ$ and $(3x + 28)^\circ$.

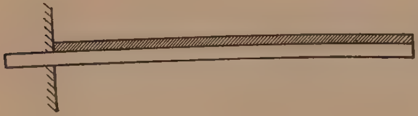
(*b*) $(\frac{3}{5}x + 16\frac{1}{2})^\circ$ and $(\frac{2}{5}x + 24\frac{1}{2})^\circ$.

(*c*) $5x^\circ$ and $(7x - 60)^\circ$.

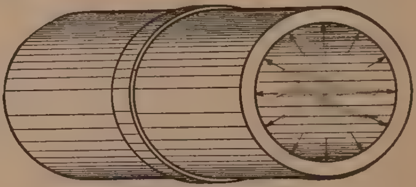
(*d*) $(3m + 15)^\circ$ and $(m + 40)^\circ$.

22. Compute the deflection at the end of a 10′ steel cantilever beam 3″ × 2″ in section when uniformly loaded with a total distributed load of 800 lbs.

$$\text{Deflection} = \frac{1}{8}\,\frac{Wl^3}{EI}.$$



Prob. 22.

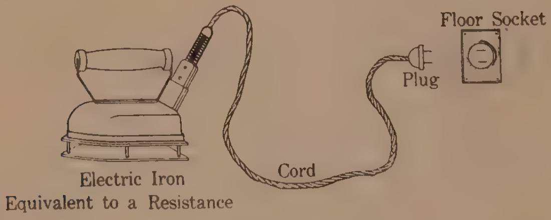


Prob. 23.

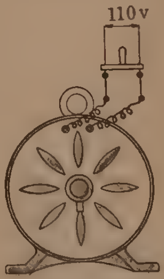
W = weight in lbs. l = length in inches. E = Modulus of elasticity or 30,000,000 pounds per square inch. I = Moment of inertia, *i.e.*, $\frac{1}{12}bd^3$ for a rectangle, where b = breadth in inches, d = depth in inches.

23. A steel cylinder 2′ in diameter is surrounded by a wrought iron hoop 2″ wide by ¼″ thick. If the radial pressure between the cylinder and the hoop is 200 pounds per square inch, what is the unit tensile stress in the hoop? $Rd = 2st$.

R = radial pressure in lbs. per sq. in., d = diameter in ins., s = stress in lbs. per sq. in., t = thickness in ins.



Prob. 24.



Prob. 25.

24. (*a*) How many calories of heat are generated in one hour by an electric iron of 220 ohms resistance using 5 amperes of current?

Formula. $H = .24I^2Rt$, where H = heat in calories, I = current in amperes, R = resistance in ohms, t = time in seconds.

(*b*) How much heat would be generated by an iron using twice as many amperes?

25. A one-quarter horse power motor operates on a 110 volt line. How much current does it consume in amperes if it is only 60% efficient?

$I = \dfrac{100 \times 746 \times H.P.}{Eff. \times E}$, where I = current in amperes, $H.P.$ = horse power, $Eff.$ = per cent efficiency, E = volt line.

CHAPTER IV

RELATIONS BETWEEN GEOMETRIC FIGURES

86. **Geometric Figures.**—A point, a line, an angle, a surface, a solid, or any combination of them, is called a geometric figure.

The relations between geometric figures form the basis of solution of many engineering problems. A problem is more easily and intelligently solved if it is first visualized by the use of a figure.

The correct solution depends upon the knowledge of the relations between the lines and the angles of the same or of related figures. Some of the fundamental relations will be studied in this chapter, while others will be taken up as the subject of mathematics is developed.

87. **Similar Figures.**—Figures which have the same shape are similar.

88. **Equivalent Figures.**—Figures which have the same size are equivalent.

89. **Equal Figures.**—Figures which have same shape and size are equal.

90. **Geometric Axioms.**—Axioms which apply to geometric figures only are called geometric axioms.

The following are important geometric axioms.

1. A straight line is the shortest line connecting two points. The distance between two points is the length of the straight line connecting them.

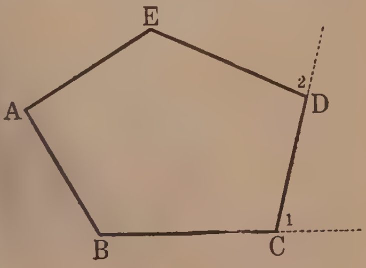
2. Through two given points only one straight line can be drawn. Hence, two points determine a straight line.

3. Two lines which have two points in common coincide throughout their whole extent. Hence, two lines can intersect in only one point.

4. Through a point not on a given line, only one line can be drawn parallel to the given line. (For definition of parallel lines see §114.)

5. A geometric figure may be moved about in space without changing its shape or size.

6. Two geometric figures which can be made to coincide are equal.



91. **Polygons.**—A polygon is a portion of a plane bounded by straight lines, as *ABCDE*. *AB*, *BC*, *CD*, etc. are the sides of the polygon. The sum of the sides is the **perimeter.** **The angles** of the polygon are the angles formed where two sides meet; as $\angle ABC$, or $\angle B$. The vertices of the angles are the **vertices** of the polygon. **Adjacent angles** of a polygon are two angles having a common side. $\angle B$ and $\angle C$ are adja-

cent and BC is the common side. **An exterior** angle of a polygon is an angle formed by one side and an adjacent side extended, as ∠ 1 and ∠ 2. A **diagonal** of a polygon is a straight line connecting any two vertices not adjacent.

92. **Equiangular Polygons.**—An equiangular polygon is a polygon all of whose angles are equal.

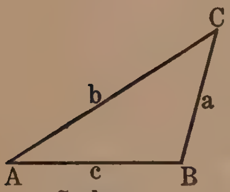
93. **Equilateral Polygons.**—An equilateral polygon is a polygon all of whose sides are equal.

94. **Triangles.**—A triangle is a polygon of three sides.

It is necessary to make a thorough study of the triangle, since it plays a very important part in the solution of problems in science and engineering, where relations between lines and between angles are required. In later chapters we shall see how to use triangles in computing heights, distances, forces, components of forces, etc.

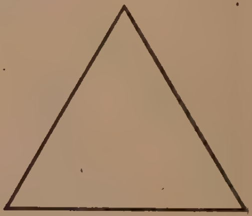
95. **Triangles Classified as to Sides.**—A **scalene** triangle has no two sides equal. An **isosceles** triangle has two equal sides. The equal sides are the **legs.** An **equilateral** triangle has three equal sides.

The sides of a triangle are often denoted by small letters, as **a, b,** and **c,** placed opposite the angles A, B, and C, respectively.

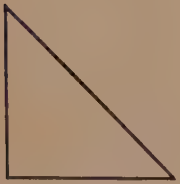


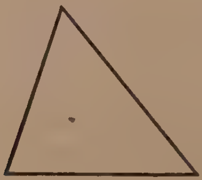
Scalene

Isosceles


Equilateral

96. **Triangles Classified as to Angles.**—A **right triangle** is a triangle in which one angle is a right angle. The side opposite the right angle is the **hypotenuse.** The other two sides are the **legs.** An **acute** triangle is one in which all angles are acute. An **obtuse triangle** is one in which one angle is obtuse. An **equiangular triangle** is one in which all angles are equal. An **oblique triangle** is a general name for all triangles except a right triangle.


Right


Acute

Obtuse

Equiangular

97. **Summary.**—The following outline will help visualize the above classification.

Triangles	As to Sides	Scalene Isosceles Equilateral	
	As to Angles	Right Oblique	Acute Obtuse

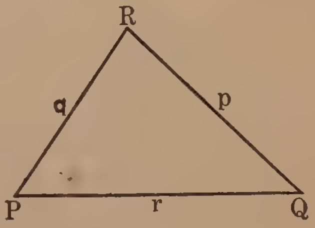
98. **Bases.**—The base of a triangle is the side upon which it is assumed to stand. Any side may be taken as the base.

99. **Angles.**—The **vertex** angle is the angle opposite the base. The other two angles are the **base angles.**

100. **Altitudes.**—An altitude of a triangle is the perpendicular drawn from a vertex to the base or to the base extended. A triangle has three altitudes since it has three bases.

101. **Medians.**—A median of a triangle is the line drawn from a vertex to the middle point of the opposite side. A triangle has three medians.

102. **Bisectors.**—A bisector of an angle of a triangle is a line which divides the angle into two equal angles. A triangle has three angle-bisectors.



103. **Properties of Triangles.**—1. The sum of any two sides of a triangle is greater than the third side.

Thus, $p + q > r$. Geom. *Ax.* 1.

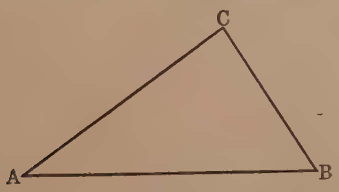
2. Any side of a triangle is greater than the difference between the other two sides.

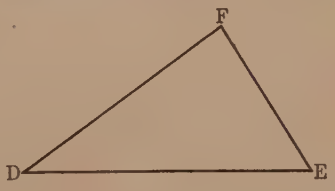
Thus, $p + q > r$ (1)

Hence, $p > r - q$. *Ax.* 9.

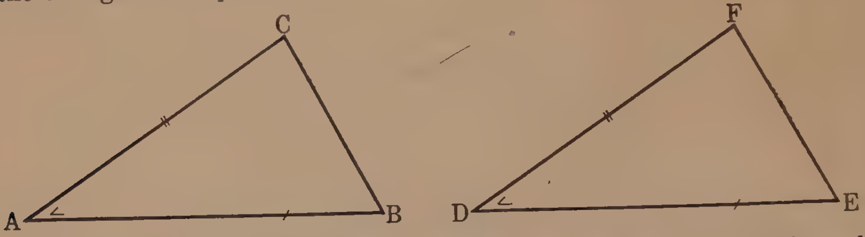
Problems. Is it possible to have a triangle whose sides are 8, 6, 12? 17, 5, 6? 11, 16, 25? 5, 12, 13? 3, 4, 5? 2.3, 3.4, 4.5? 3.5, 4.6, 9?

104. **Corresponding Parts of Equal Triangles.**—The corresponding parts of two triangles are the parts similarly placed in the two triangles. Corresponding sides are always opposite equal angles. Thus, in the two triangles *ABC* and *DEF*, *AB* corresponds to *DE*, *BC* to *EF*, *AC* to *DF*, $\angle A$ to $\angle D$, $\angle B$ to $\angle E$, and $\angle C$ to $\angle F$. Corresponding parts of equal triangles are equal.





105. **Theorem.**—If two sides and the included angle of one triangle are equal to the corresponding two sides and the included angle of another, the triangles are equal.



Given △s ABC and DEF, in which $AB = DE$, $AC = DF$, and $\angle A = \angle D$.

To prove that $\triangle ABC = \triangle DEF$.

Proof.—Place the $\triangle ABC$ on the $\triangle DEF$ so that the side AB coincides with its equal DE. §90—5.

Then AC will take the direction of DF, for $\angle A = \angle D$. Given.

Then C will fall on F, for $AC = DF$. Given.

∴ BC coincides with EF.

(Through two points only one straight line can be drawn.)

∴ △s ABC and DEF coincide and are equal. §90—6.

Q.E.D.

Name the corresponding parts of the above triangles.

The above theorem may be abbreviated thus, s. ∠. s.

This method of proof is known as the "proof by superposition."

106. **Corollary.**—A corollary is a truth easily deduced from one or more theorems previously proven.

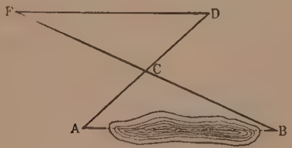
107. **Corollary** 1.—If the legs of one right triangle are equal to the legs of another, the two right triangles are equal.

Proof.—Since all right angles are equal, we have two triangles with two sides and the included angle of one equal to the corresponding two sides and the included angle of the other.

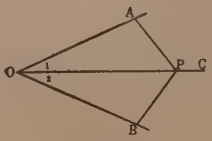
∴ The two *rt* △s are equal.

PROBLEMS

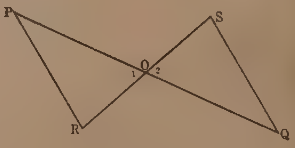
1. Find the distance between two points A and B, when an obstacle stands between them. Take C any convenient point. Measure AC and extend it to D, making $CD = AC$. Measure BC and extend it to E, making $CE = BC$. Measure ED. Then $AB = ED$. §105.



PROB. 1.



PROB. 3.



PROB. 5.

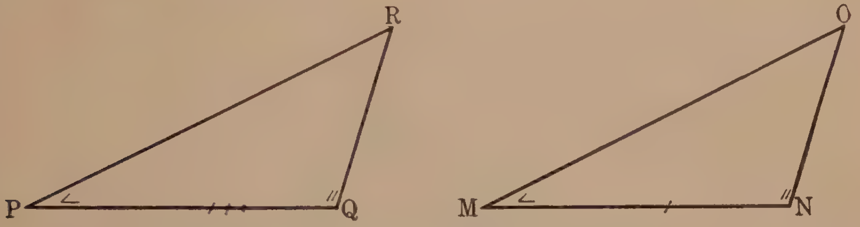
2. Construct $\triangle PQR$ with side PQ = side QR. Extend PQ through Q to T, and RQ to S, making $SQ = QT$. Prove that $\triangle PQS = \triangle QRT$.

3. Given $\angle O$ bisected by OC, P any point in OC and $OA = OB$. Prove $AP = BP$.

4. Any point in the bisector of the vertex angle of an isosceles triangle is equidistant from the extremities of the base.

5. Given lines PQ and RS bisecting each other at O. Prove that $PR = SQ$.

108. **Theorem.**—If two angles and the included side of one triangle are equal to the corresponding two angles and the included side of another, the triangles are equal.



Given $\triangle$s PQR and MNO in which $\angle P = \angle M$, $\angle Q = \angle N$, and $PQ = MN$.

To prove that $\triangle PQR = \triangle MNO$.

Proof.—Place the $\triangle PQR$ on the $\triangle MNO$ so that PQ coincides with its equal MN. § 90—5

Then PR will take the direction of MO and R will fall on MO or on MO extended, for $\angle P = \angle M$. Given.

Again, QR will take the direction of NO and R will fall on NO or NO extended, for $\angle Q = \angle N$. Given.

Since R falls on MO and NO it will fall on O. (Two straight lines can intersect in only one point.)

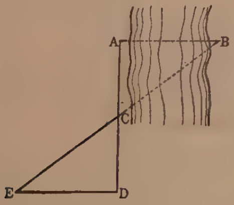
$\therefore$ $\triangle$s PQR and MNO coincide and are equal. *Q.E.D.*

The abbreviation for this theorem is $\angle$. s. $\angle$.

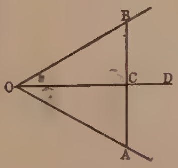
Name the corresponding parts of the above triangles.

PROBLEMS

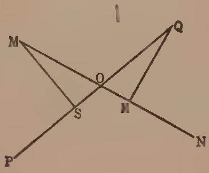
1. Find the distance between two points, A and B, on opposite sides of a stream. At A lay off an angle of 90°. Measure AC and extend it to D, making $CD = AC$. At D lay off a perpendicular meeting BC extended at E. Measure ED. Prove that $ED = AB$. §108.



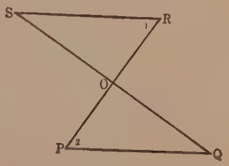
Prob. 1.



Prob. 2.



Prob. 3.



Prob. 4.

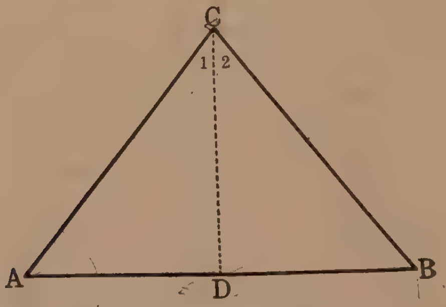
2. Given $\angle O$ bisected by OD, and $AB \perp OD$ at C. Prove that $OB = OA$.

3. Given lines MN and PQ intersecting at O, $MS \perp PQ$, $QR \perp MN$, and $OS = OR$. Prove that $\angle M = \angle Q$, and $OM = OQ$.

4. PR and SQ were drawn so that $PO = OR$, and $\angle 1 = \angle 2$. Show that $SR = PQ$ and $SO = OQ$.

109. **Auxiliary Lines.**—To prove a theorem it is often necessary to draw additional lines in the given figure. These lines reveal properties that are shown in the proofs of previous theorems, and thus aid in establishing the required proof. Such lines are **auxiliary** lines and are always dotted to distinguish them from the given lines.

110. **Theorem.**—In an isosceles triangle the angles opposite the equal sides are equal.



Given the isosceles $\triangle$ ABC in which $CA = CB$.
To prove that $\angle A = \angle B$.
Proof.—Draw CD bisecting $\angle C$.
Then in $\triangle$s ADC and DBC,

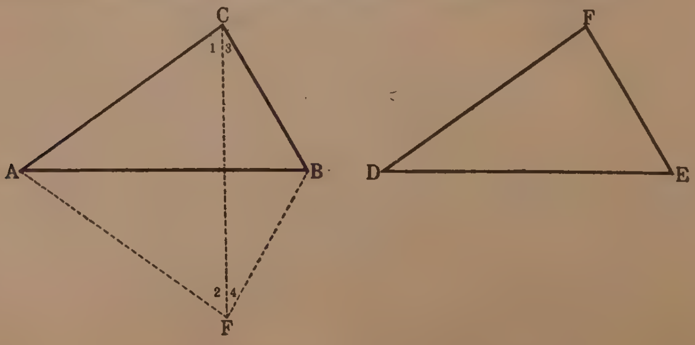
$CD = CD$ Common.
$\angle 1 = \angle 2$, Const.
and $CA = CB$. Given.
$\therefore \triangle ADC = \triangle DBC$, s. $\angle$. s.
$\therefore \angle A = \angle B$. Cor. $\angle$s of equal $\triangle$s.
Q.E.D.

Name other corresponding parts of the $\triangle$s ADC and DBC.

PROBLEMS

1. Prove that the medians to the legs of an isosceles triangle are equal.
2. If two rafters of a gable roof are of equal length, supports perpendicular to the floor, set at equal distances from the center line, are equal.
3. Prove that the bisectors of the base angles of an isosceles triangle are equal.
4. An equilateral triangle is equiangular.

111. **Theorem.**—If the three sides of one triangle are equal to the corresponding three sides of another, the triangles are equal.



Given $\triangle$s ABC and DEF in which $AB = DE$, $BC = EF$, and $CA = FD$.

To prove that $\triangle ABC = \triangle DEF$.

Proof.—Place the triangle DEF so that DE will coincide with its equal AB and $\triangle DEF$ takes the position ABF.

Draw CF. Then $\triangle$s AFC and FBC are isosceles. Why?
$\angle 1 = \angle 2$, Why?
and $\angle 3 = \angle 4$. Why?
$\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$. Why?
i.e., $\angle ACB = \angle AFB$. *Ax.* 12.
$\therefore \triangle ABC = \triangle AFB$. §105.
$\therefore \triangle ABC = \triangle DEF$. *Ax.* 5.
The abbreviation for this theorem is s.s.s. *Q.E.D.*

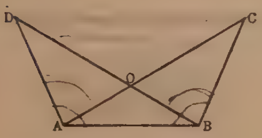
Name the corresponding $\angle$s of the two $\triangle$s.

PROBLEMS

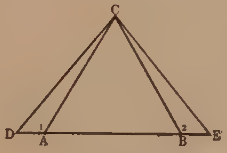
1. ABC and ABD are isosceles triangles on the same base AB. Prove that CD bisects the angles C and D. (C and D are on opposite sides of the base AB.)

2. Given $AD = BC$ and $AC = BD$. Prove that $\angle DAB = \angle ABC$, $\angle DAO = \angle CBO$, and $\triangle AOD = \triangle BOC$.

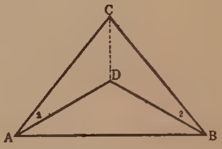
3. Given isosceles $\triangle ABC$ and AB extended so that $AD = BE$. Prove that $\angle 1 = \angle 2$ and $DC = CE$.



PROB. 2.



PROB. 3.

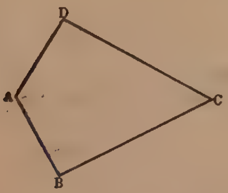


PROB. 4.

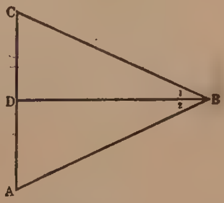
4. Given isosceles $\triangle$s ABC and ABD on the same base AB. Prove that $\triangle ADC = \triangle DBC$, and $\angle 1 = \angle 2$.

5. Given $AD = AB$ and $DC = BC$. Prove that $\angle D = \angle B$.

6. If $AB = CB$ and D is the midpoint of AC, prove that $\angle 1 = \angle 2$, and $\angle C = \angle A$.



PROB. 5.



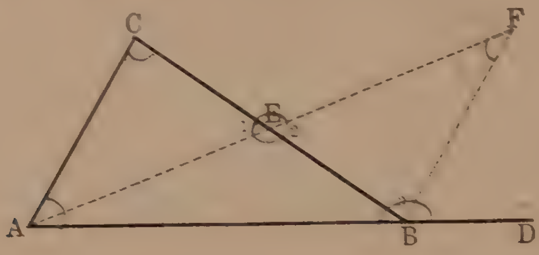
PROB. 6.

112. Theorem.—An exterior angle of a triangle is greater than either opposite interior angle.

Given $\angle CBD$ an exterior angle of $\triangle ABC$.

To prove that $\angle CBD > \angle C$ or $\angle A$.

Proof.—Take E the mid-point of CB. Draw AE and continue it to F, making $EF = AE$. Draw FB.



Then $\triangle AEC = \triangle EBF$. §105.

$\therefore \angle C = \angle 3$. §104.

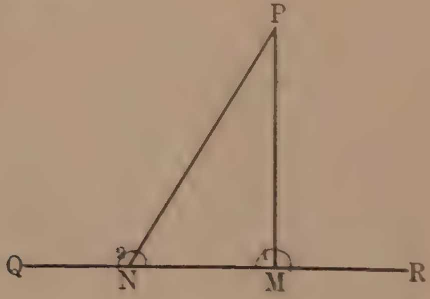
$\angle CBD > \angle 3$. *Ax.* 12.

$\therefore \angle CBD > \angle C$. *Ax.* 5.

Q.E.D.

To prove $\angle CBD > \angle A$, take the mid-point of AB and draw a figure similar to the one above. The student should complete the proof.

113. **Theorem.**—From a point without a line only one perpendicular can be drawn to the line.



Given $PM \perp QR$ and PN any other line from P to QR.

To prove that PN is not $\perp QR$.

Proof. $\angle 2 > \angle 1$. §112.

$\angle 1$ is a rt $\angle$. Given.

$\therefore \angle 2$ is obtuse. Why?

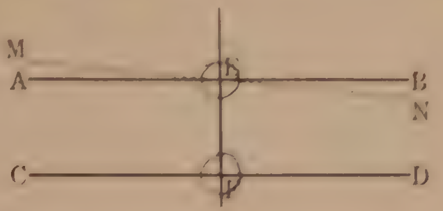
$\therefore PN$ is not $\perp QR$. §75.

Q.E.D.

114. **Parallel Lines.**—Parallel lines are lines in the same plane which do not meet however far extended.

115. **Theorem.**—Two straight lines $\perp$ to the same straight line are $\parallel$. §113, 114.

116. **Theorem.**—If a line is perpendicular to one of two parallel lines it is perpendicular to the other also.

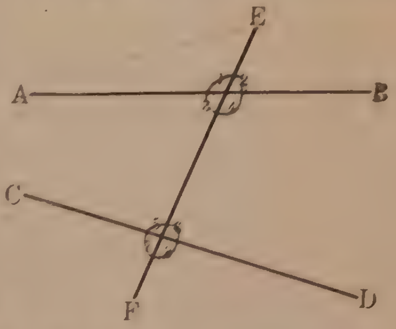


Given $AB \parallel CD$, $EF \perp CD$.
To prove that $EF \perp AB$.
Proof.—Through E draw $MN \perp$ to EF.
Then $MN \parallel CD$. §115.
But $AB \parallel CD$. Given.
$\therefore MN$ coincides with AB. §90—4.
But $MN \perp EF$. Const.
$\therefore AB \perp EF$. (AB coincides with MN which is $\perp EF$.)
i.e., $EF \perp AB$. Q.E.D.

117. **Corollary.**—If two straight lines are parallel to a third straight line they are parallel to each other.

118. **Definitions.**—A **transversal** is a straight line which intersects two or more straight lines.

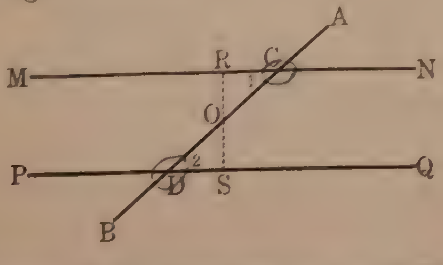
Thus, EF intersects AB and CD, forming eight angles as in the figure.



Angles 1, 2, 7, and 8 are **exterior angles**. Angles 3, 4, 5, 6 are **interior angles**. Angles 3 and 6; 4 and 5 are pairs of **alternate interior angles.**

Angles 1 and 5; 2 and 6; 3 and 7; 4 and 8 are pairs of **corresponding angles.**

119. **Theorem.**—If two parallel lines are cut by a transversal, the alternate interior angles are equal.



Given $MN \parallel PQ$ and AB a transversal cutting MN in C and PQ in D.
To prove that $\angle 1 = \angle 2$.

Proof.—Through O, the mid-point of CD, draw $RS \perp PQ$. Then $RS \perp MN$. §116.

Apply the figure DSO to the figure CRO so that OD coincides with its equal OC.

Point D will fall on point C.

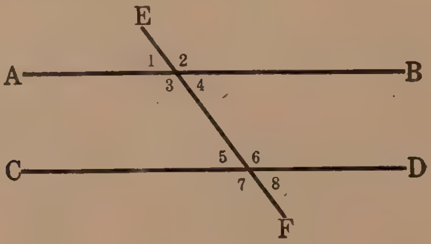
Then OS will take the direction of OR. (Vert. $\angle$s at O are equal.)

DS will fall along CR. §113.

$\therefore \angle 1 = \angle 2$. §90–6.

Q.E.D.

120. **Corollary 1.**—If two parallel lines are cut by a transversal any pair of corresponding angles are equal.



Suggestion.

$AB \parallel CD$, EF is a transversal.

$\angle 1 = \angle 4$. Why?

$\angle 4 = \angle 5$. §119.

$\therefore \angle 1 = \angle 5$. *Ax.* 6.

Prove that $\angle 2 = \angle 6$, $\angle 4 = \angle 8$, and $\angle 3 = \angle 7$.

121. **Corollary 2.**—If two parallel lines are cut by a transversal the two interior angles on the same side of the transversal are supplementary.

PROBLEMS

1. Draw a pair of parallel lines cut by a transversal. If a pair of alternate interior angles are $(4x - 6)^\circ$ and 50°, find the value of x and of the eight angles of the figure.

2. If a pair of corresponding angles are $(\frac{1}{2}x + 15)^\circ$ and 25°, find x and the eight angles of the figure.

3. If the two interior angles on the same side of the transversal are $(\frac{2}{3}x + 40)^\circ$ and 100°, find x and all the angles of the figure.

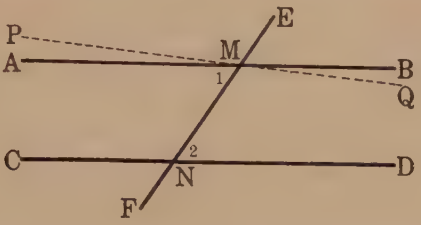
122. **Converse of a Theorem.**—The converse of a theorem is the theorem formed by interchanging the hypothesis and the conclusion.

Illustration.—If two straight lines are cut by a transversal making the alternate interior angles equal, the straight lines are parallel. This is the converse of the theorem in §119.

It does not follow, however, that the converse of a theorem is true because the theorem is true. A simple statement will make this clear.

All right angles are equal. The converse, all equal angles are right angles, is **not** true.

123. **Theorem.**—If two straight lines are cut by a transversal making the alternate interior angles equal, the lines are parallel.



Given AB and CD two straight lines cut by the transversal EF in the points M and N, making $\angle 1 = \angle 2$.

To prove that AB and CD are parallel.

Proof.—Through M draw a line PQ parallel to CD.

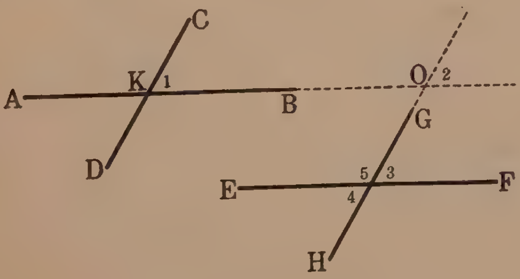
Then $\angle PMN = \angle 2$. §119.
But $\angle 1 = \angle 2$. Given.
$\therefore \angle PMN = \angle 1$. *Ax.* 6.
$\therefore PM$ and AM coincide. §90–4.
$\therefore PQ$ and AB coincide. §90–3. Geom. *Ax.* 3.

Since PQ was constructed parallel to CD, AB which coincides with PQ is parallel to CD. *Q.E.D.*

124. **Corollary** 1.—If two straight lines are cut by a transversal making any pair of corresponding angles equal, the lines are parallel.

125. **Corollary** 2.—If two straight lines are cut by a transversal making the interior angles on the same side of the transversal supplementary, the lines are parallel.

126. **Theorem.**—If the corresponding sides of two angles are parallel, the angles are either equal or supplementary.



Given $AB \parallel EF$ and $CD \parallel GH$.

To prove that $\angle 1 = \angle 3 = \angle 4$; $\angle 1$ and $\angle 5$ supplementary.

Proof.—Extend AB and HG to intersect at O.

Give a reason for each of the following:

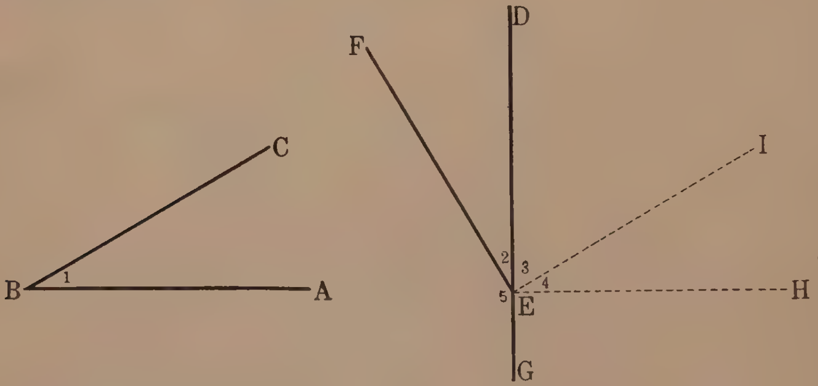
$\angle 1 = \angle 2.$
$\angle 2 = \angle 3.$
$\therefore \angle 1 = \angle 3.$
$\angle 3 = \angle 4.$
$\therefore \angle 1 = \angle 3 = \angle 4.$
$\angle 5$ is the supplement of $\angle 3$.
$\therefore \angle 1$ is the supplement of $\angle 5$.

Q.E.D.

Note.—*From the above it is seen that when both pairs of corresponding sides extend in the same direction or in opposite directions from the vertices, the angles are equal.*

When one pair of the corresponding sides extends in the same direction and the other pair in opposite directions from the vertices, the angles are supplementary.

127. **Theorem.**—If the sides of one angle are perpendicular to the corresponding sides of another angle, the angles are either equal or supplementary.



Given $BA \perp DG$ and $BC \perp FE$.

To prove that $\angle 1 = \angle 2$; $\angle 1$ and $\angle 5$ supplementary.

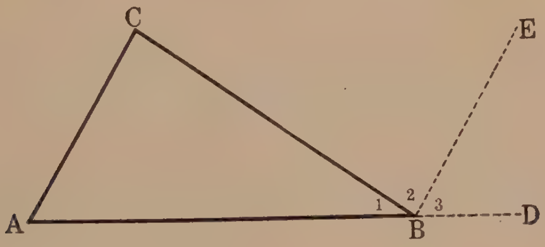
Proof.—Draw $EH \perp DG$ and $EI \perp EF$, extending in the same directions as BA and BC respectively.

Then $EH \parallel BA$ and $EI \parallel BC$.	§115.
$\therefore \angle 1 = \angle 4.$	§126.
$\angle 4$ is the complement of $\angle 3$.	Const.
$\angle 2$ is the complement of $\angle 3$.	Const.
$\therefore \angle 4 = \angle 2.$	§82.
But $\angle 1 = \angle 4$, $\therefore \angle 1 = \angle 2.$	*Ax.* 6.
$\angle 2$ is the supplement of $\angle 5$.	Why?
$\therefore \angle 1$ is the supplement of $\angle 5$.	*Ax.* 5.

Q.E.D.

Note.—*In the above theorem two angles are equal if the corresponding sides are considered to rotate about the vertex in the same direction. Thus,* *BA rotates opposite to the hands of a clock to meet BC, and ED rotates in the same direction as BA to meet EF.* ∴ ∠ 1 = ∠ 2. *The angles are supplementary if the corresponding sides rotate in opposite directions. Thus, BA rotates opposite to the hands of a clock to meet BC, but EG rotates in the same direction as the hands of a clock to meet EF, therefore,* ∠ 1 *and* ∠ 5 *are supplements.*

128. **Theorem.**—The sum of the angles of any triangle equals two right angles.



Given *ABC* any △.
To prove that ∠ 1 + ∠ *C* + ∠ *A* = 2 *rt.* ∠s.
Proof.—Extend *AB* to *D* forming the exterior ∠ *CBD*.
Draw *BE* ∥ *AC*.
(1) Then ∠ 1 + ∠ 2 + ∠ 3 = 2 *rt.* ∠s. Why?
But ∠ *C* = ∠ 2. (Alt-int ∠s of ∥ lines.)
∠ *A* = ∠ 3. (Cor. ∠s of ∥ lines.)
In (1), substitute ∠ *C* for ∠ 2 and ∠ *A* for ∠ 3.
Then ∠ 1 + ∠ *C* + ∠ *A* = 2 *rt.* ∠s. *Q.E.D.*

Prove the above theorem by drawing a line through a vertex parallel to the opposite side.

129. **Corollary** 1.—An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

130. **Corollary** 2.—There can be but one right angle in a triangle.

131. **Corollary** 3.—The sum of the two acute angles of a right triangle equals one right angle.

132. **Corollary** 4.—If an acute angle of one right triangle is equal to an acute angle of another right triangle, the remaining acute angles are equal.

133. **Corollary** 5.—There can be but one obtuse angle in a triangle.

134. **Corollary** 6.—If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.

135. **Corollary** 7.—Two triangles are equal if two angles and a side of one are equal to two angles and the corresponding side of the other.

136. **Corollary** 8.—Two right triangles are equal if an acute angle and a leg of one are equal to an acute angle and a corresponding leg of the other.

137. **Corollary** 9.—Two right triangles are equal if an acute angle and the hypotenuse of one are equal to an acute angle and the hypotenuse of the other.

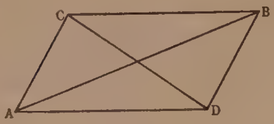
138. **Corollary** 10.—The sum of the exterior angles of a triangle formed by producing the sides in order equals 4 *rt.* ∠s. Prove by equations.

139. **Corollary** 11.—Each acute angle of an isosceles right triangle is 45°.

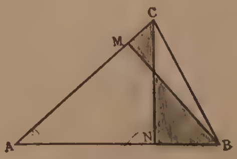
140. **Corollary** 12.—Each angle of an equiangular triangle is 60°.

PROBLEMS

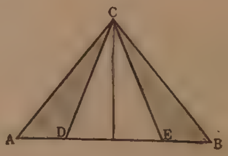
1. Two angles of a triangle are 42° and 62°. Find the other angle. Find all the exterior angles.
2. One acute angle of a right triangle is 38° 42′. Find the other acute angle.
3. How many degrees in each angle of an isosceles right triangle?
4. Is it possible to have a triangle whose angles are 30° 15′, 48° 12′, 101° 33′? 26° 37′, 46° 25′, 107° 1′?
5. If the vertex angle of an isosceles triangle is 30°, find the base angles.
6. A base angle of an isosceles triangle is 55°. Find the vertex angle.
7. The vertex angle of an isosceles triangle is 45°. Find the three exterior angles.
8. An exterior angle at the base of an isosceles triangle is 128° 20′. Find all the angles of the triangle.
9. If the acute angles of a right triangle are equal, how many degrees in each angle?
10. If one acute angle of a right triangle is double the other, how many degrees in each angle?
11. The second angle of a triangle is 25° more than the first, the third angle is 10° more than the second. Find the three angles.
12. The base angles of a triangle are 75° and 80°. Find the angle included by their bisectors.
13. The first angle of a triangle is 30° and the second is double the third. Find all the angles. Find all the exterior angles.
14. The angles of a triangle are represented by 1, 2, and 3. If ∠ 1 + ∠ 3 = 100° and ∠ 2 + ∠ 3 = 120°, find all the angles.
15. Any line drawn parallel to the base of an isosceles triangle makes equal angles with the legs or the legs extended through the vertex.
16. Find the angles of a triangle if the second angle is 10° more than twice the first and the third angle is 40° more than twice the first.
17. An exterior angle at the base of an isosceles triangle is three times the base angle. Find the vertex angle.
18. The vertex angle of an isosceles triangle is $\frac{5}{9}$ of a right angle. Find the base angles and all exterior angles.
19. If the lines *AB* and *CD* bisect each other, prove that *CB* ∥ *AD* and *CA* ∥ *BD*.



PROB. 19.



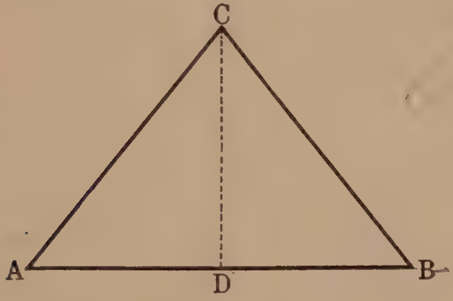
PROB. 20.



PROB. 21.

20. *ABC* is any triangle. *BM* and *CN* are altitudes. Name pairs of equal angles on the figure.
21. In the isosceles triangle *ABC*, *AD* = *EB*. Prove that △ *DEC* is isosceles.

141. Theorem.—If two angles of a triangle are equal, the triangle is isosceles. Converse of §110.

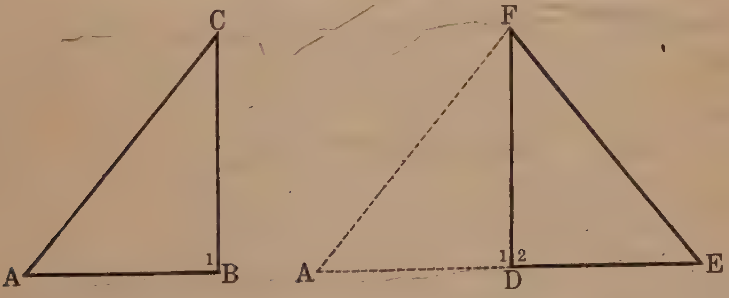


Given $\triangle ABC$ in which $\angle A = \angle B$.
To prove that $CA = CB$.
Proof.—Draw $CD \perp$ to AB.
Then in rt. $\triangle$s ADC and CDB,

$\angle A = \angle B$,	Given.
and $CD = CD$.	Common.
$\therefore$ rt. $\triangle ADC =$ rt. $\triangle CBD$.	§136.
	Q.E.D.

CA - CB -

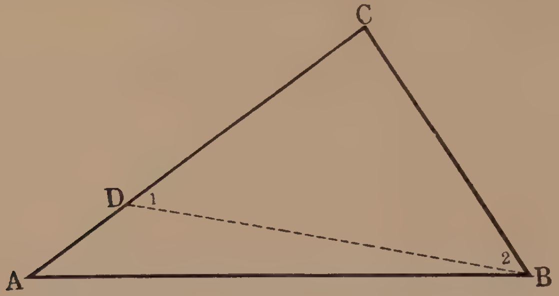
142. Theorem.—Two right triangles are equal, if the leg and hypotenuse of one equal the corresponding leg and hypotenuse of the other.



Given rt. $\triangle$s ABC and DEF in which $AC = FE$ and $CB = FD$.
To prove that rt. $\triangle ABC =$ rt. $\triangle DEF$.
Proof.—Place the rt. $\triangle ABC$ so that CB coincides with its equal FD and the $\triangle ABC$ takes the position FDA.
Now ADE is a straight line, since $\angle 1 + \angle 2 =$ a straight angle.
Why is it necessary to show that ADE is a straight line?

Then AEF is an isosceles triangle.	Why?
$\therefore \angle A = \angle E$.	Why?
$\therefore$ rt. $\triangle ABC =$ rt. $\triangle DEF$.	§137.
	Q.E.D.

143. **Theorem.**—If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.



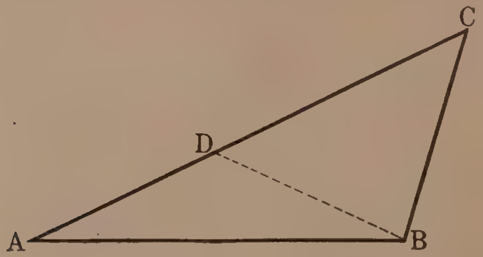
Given $\triangle ABC$ in which $AC > CB$.
To prove that $\angle B > \angle A$.
Proof.—On CA lay off $CD = CB$.
Draw DB.

Then $\angle B > \angle 2$	*Ax.* 13.
$\angle 1 = \angle 2$	Why?
$\angle B > \angle 1$	*Ax.* 5.
But $\angle 1 > \angle A$	§112.
$\therefore \angle B > \angle A$	

(If of three quantities, the first is greater than the second and the second greater than the third, then the first is greater than the third.)

Q.E.D.

144. **Theorem.**—If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.



Given the $\triangle ABC$ in which $\angle B > \angle A$.
To prove that $AC > BC$.
Proof.—Draw BD making $\angle ABD = \angle A$.

Then, $DA = DB$.	§141.
$BD + DC > BC$.	§103.
$AD + DC > BC$.	*Ax.* 5.
$\therefore AC > BC$.	*Ax.* 12.

Q.E.D.

Another form of proof is the **indirect method,** by which the conclusion is arrived at by the process of elimination.

Thus, CA equals CB, is less than CB, or is greater than CB. (When two quantities are compared, the first $=$, $>$, or $<$ the second.)

First.—Suppose $CA = CB$

Then $\triangle ABC$ is isosceles and $\angle A = \angle B$. §110.

This is contrary to the hypothesis.

Second.—Suppose $CA < CB$. Then $\angle A > \angle B$. §143.

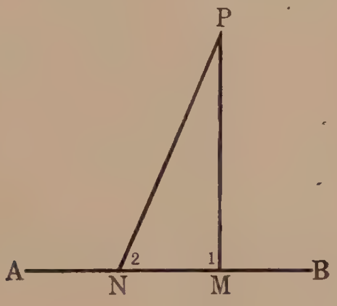
This is contrary to the hypothesis.

$\therefore CA > CB$.

For if CA is neither equal to, nor less than CB it must be greater.

Q.E.D.

145. **Theorem.**—The perpendicular is the shortest line that can be drawn from a point to a line.



Given $PM \perp AB$ and PN any line not $\perp AB$.

To prove that $PM < PN$.

Proof.— $\angle 1$ is a rt. $\angle$. Given.

$\angle 2$ is an acute $\angle$. §130.

$\therefore PM < PN$. §144.

Q.E.D.

Is the converse of the above theorem true? Prove your answer.

146. **Theorem.**—If two triangles have two sides of the one equal to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first triangle is greater than the third side of the second. Illustrate this fact.

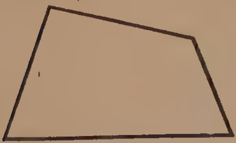
A rigorous proof of this theorem may be given if desired.

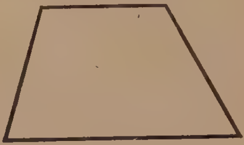
The converse of the above theorem is true.

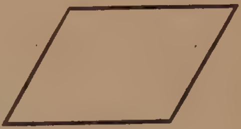
147. **Polygons Classified According to the Number of Sides.**—A polygon of three sides is a **triangle;** one of four sides, **quadrilateral;** one of five sides, a **pentagon;** one of six sides, a **hexagon;** one of eight sides, an **octagon;** one of ten sides, a **decagon,** etc.

148. **Quadrilaterals Classified.**—A **trapezium** is a quadrilateral with no two sides parallel. A **trapezoid** is a quadrilateral with two sides parallel. The parallel sides are the **bases** of the trapezoid. The non-

parallel sides are the **legs**. An **isosceles trapezoid** is one whose two legs are equal. The **median** is a line joining the mid-points of the legs. The **altitude** is the perpendicular distance between the bases.

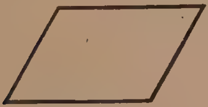

Trapezium

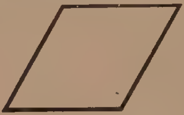

Trapezoid

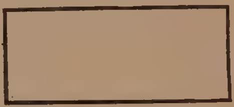

Parallelogram

A **parallelogram** is a quadrilateral whose opposite sides are parallel. The **altitude** is the perpendicular distance between the bases.

149. **Parallelograms Classified.**—A **rhomboid** is an oblique angled parallelogram. A **rhombus** is an equilateral rhomboid. A **rectangle** is a right angled parallelogram. A **square** is an equilateral rectangle.

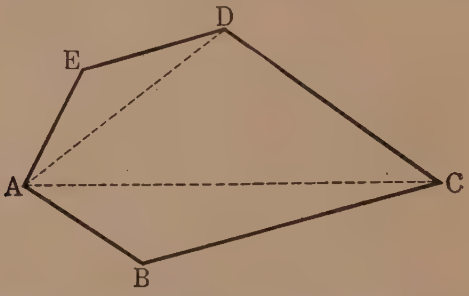

Rhomboid


Rhombus


Rectangle

Square

150. **Theorem.**—The sum of the angles of a polygon of **n** sides is equal to $(n - 2)$ 2 rt. $\angle$s.



Given ABC etc. a polygon of n sides. (Here $n = 5$.)

To prove that $\angle A + \angle B + \angle C$ etc. $= (n - 2)$ 2 rt. $\angle$s.

Proof.—If all the diagonals are drawn from any one vertex, the polygon will be divided into $(n - 2)$ triangles.

The sum of the $\angle$s of any triangle equals 2 rt. $\angle$s.

The sum of the angles of the triangles equals the sum of the angles of the polygon. *Ax.* 12.

$\therefore \angle A + \angle B + \angle C$ etc. $= (n - 2)$ 2 rt. $\angle$s.

Q.E.D.

Prove this theorem by taking any point within the polygon and drawing lines from this point to the vertices. How many triangles will be formed?

151. **Corollary** 1.—Each angle of an equiangular polygon of **n** sides equals $\frac{(n-2)\,2 \text{ rt } \angle\text{s}}{n}$.

152. **Corollary** 2.—The sum of the exterior angles of a polygon of **n** sides equals 4 rt. ∠s.

153. **Corollary** 3.—Each exterior angle of an equiangular polygon equals $\frac{4 \text{ rt. } \angle\text{s}}{n}$.

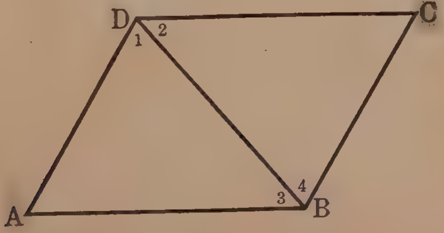
PROBLEMS

1. The angles of a triangle are $5x°$, $20x°$ and 30°. Find them.
2. The angles of a triangle are in the same relation as 2, 3, and 4. Find them.
Hint. *Let* x = *an* ∠ *common to the three* ∠*s. Then* $2x$ etc. = *the* ∠*s.*
3. The angles of a triangle are $y°$, $(2y + 10)°$, and $(3y - 15)°$. Find them.
4. The acute angles of a right triangle are $x°$ and $2x°$. Find them.
5. The acute angles of a right triangle are $2x°$ and $\frac{x°}{4}$. Find them.
6. One acute angle of a right triangle is seven times the other. Find them.
7. An exterior angle of a right triangle is 127°. Find the two acute angles of the triangle.
8. The second angle of a triangle is 25° more than twice the first, the third is 10° less than three times the second. Find the angles of the triangle.
Ans. 10°, 45°, 125°.
9. Find an angle whose supplement plus 20° is $\frac{3}{2}$ of its complement.
Ans. −130°.
10. The difference between the acute angles of a right triangle is 20°. Find them.
11. The difference between the acute angles of a right triangle is $b°$. Find them.
12. The vertex angle of an isosceles triangle is 10° less than $\frac{5}{9}$ of a right angle. Find the angles. $\frac{44}{40}$
13. Find the angles of an isosceles triangle if the vertex angle is two and one-half times a base angle.
14. Find the angles of an isosceles triangle if the vertex angle is 15° less than a base angle.
15. How many sides has a polygon if the sum of its angles is 8 right angles?
Ans. 6 sides.
16 How many sides has a polygon if the sum of its angles is 26 right angles?
17. How many degrees in each angle of the following equiangular polygons: Triangle; Quadrilateral; Hexagon.

154. **Theorem.**—A diagonal divides a parallelogram into two equal triangles.

Given $ABCD$ a parallelogram and DB a diagonal.

To prove that △ ABD = △ BCD.



Proof.—In $\triangle ABD$ and BCD,

$DB = DB$,

$\angle 1 = \angle 4$, Why?

and $\angle 3 = \angle 2$. Why?

$\therefore \triangle ABD = \triangle BCD$. *Q.E.D.*

155. **Corollary** 1.—The opposite sides of a parallelogram are equal.

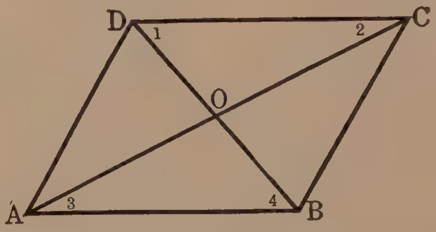
156. **Corollary** 2.—The opposite angles of a parallelogram are equal.

157. **Corollary** 3.—Parallel lines included between parallel lines are equal.

158. **Corollary** 4.—Parallel lines are everywhere equidistant.

Does the diagonal of a parallelogram bisect the angles of the parallelogram?

159. **Theorem.**—The diagonals of a parallelogram bisect each other.



Given ▱ $ABCD$ in which diagonals AC and DB intersect at O.

To prove that $AO = OC$ and $DO = OB$.

Proof.—In the $\triangle$s AOB and DOC,

$AB = DC$, Why?

$\angle 3 = \angle 2$, Why?

and $\angle 4 = \angle 1$. Why?

$\therefore \triangle AOB = \triangle DOC$. Why?

$\therefore AO = OC$ and $BO = OD$ Why?

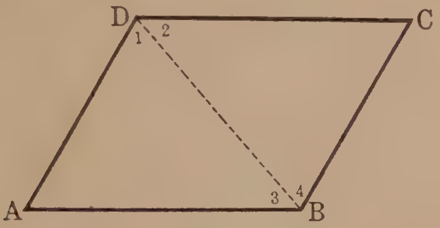
Q.E.D.

PROBLEMS

1. Are the diagonals of a parallelogram equal? Prove your answer.
2. Prove that if the diagonals of a quadrilateral bisect each other the figure is a parallelogram.
3. The diagonals of a rectangle are equal.
4. State and prove the converse of problem 3.
5. The diagonals of a square bisect each other at right angles and bisect the angles of the square.
6. The diagonals of a rhombus bisect each other at right angles and bisect the angles of the rhombus.
7. Any two adjacent angles of a parallelogram are supplementary.
8. One of two adjacent angles of a ▱ is $\frac{2}{3}$ the other. Find all the angles of the ▱.
9. One angle of a ▱ is 108° Find the other angles.
10. The bisectors of any two adjacent angles of a parallelogram are perpendicular to each other.
11. The bisectors of two opposite angles of a parallelogram are parallel.

Wed

160. Theorem.—If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.



Given quadrilateral $ABCD$ in which $AB = DC$ and $AD = BC$.

To prove that $ABCD$ is a parallelogram.

Proof.—Draw the diagonal DB.

Then $\triangle ABD = \triangle BCD$. §111.
$\therefore \angle 1 = \angle 4$. Cor. $\angle$s of $= \triangle$s.
$\therefore AD \parallel BC$. §123.
Also $\angle 3 = \angle 2$. Why?
$\therefore AB \parallel DC$. Why?
$\therefore ABCD$ is a parallelogram

A parallelogram is a quadrilateral whose opposite sides are parallel. *Q.E.D.*

161. Theorem.—If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

Given quadrilateral $ABCD$ in which $AB = DC$ and $AB \parallel DC$.

To prove that $ABCD$ is a parallelogram.

Supply the proof using the above figure.

162. Theorem.—If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on any transversal.

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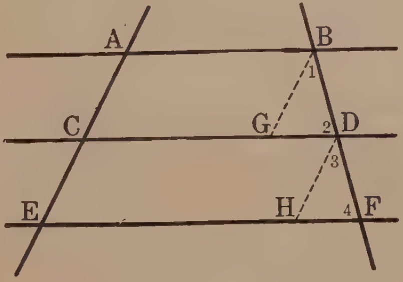


Given AB, CD, and EF parallel lines, intercepting transversal BF so that $BD = DF$.

To prove that $AC = CE$.

Proof.—Draw BG and DH parallel to AE.

Then $BG \parallel DH$ §117.

In $\triangle$s BGD and DHF,

$\angle 1 = \angle 3,$ Why?

$\angle 2 = \angle 4,$ Why?

and $BD = DF.$ Why?

$\therefore \triangle BGD = \triangle DHF.$ §108.

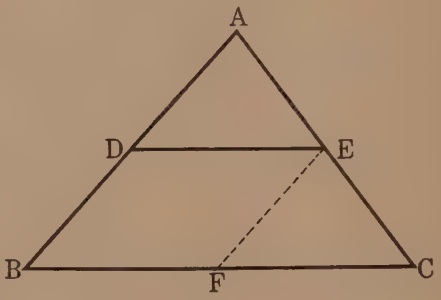
$\therefore BG = DH.$ §104.

But $AC = BG$ and $CE = DH.$ §157.

$\therefore AC = CE.$ *Ax.* 7.

Q.E.D.

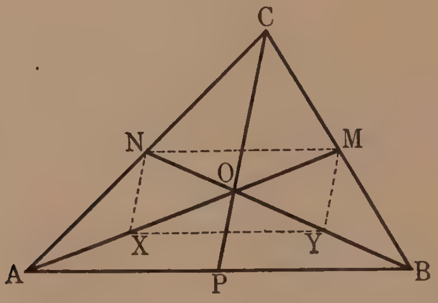
163. **Corollary.**—A line parallel to the base of a triangle and bisecting one side of the triangle bisects the other side and is equal to half the base.



Hint.—*Through A draw a line* $\parallel$ *BC. From E the mid-point of AC, draw EF* $\parallel$ *AB.*

164. **Corollary 2.**—A line parallel to a base of a trapezoid bisecting one leg, bisects the other leg, and is equal to one-half the sum of the bases.

165. **Theorem.**—The medians of a triangle meet in a point two-thirds the distance from each vertex to the middle of the opposite side.



Given AM, BN, and CP the medians of $\triangle ABC$.

To prove that AM, BN, and CP pass through O and $AO = 2OM$, etc.

Proof.—Let AM and BN intersect at O.

Take X and Y the mid-points of AO and BO.

Draw NX, XY, YM, and MN.

Then, in $\triangle ABO$, $XY \parallel AB$ and $= \frac{1}{2}AB$. §163.

In $\triangle ABC$, $NM \parallel AB$ and $= \frac{1}{2}AB$. Why?

$\therefore XY \parallel NM$ and $= NM$. Why?

$\therefore$ Figure $NXYM$ is a parallelogram. §161.

$AX = XO = OM$. Also $BY = YO = ON$. Const. and §159.

Follow the same proof by letting BN and CP intersect at O'. Prove that O and O' are the same point. Then AM, BN, and CP intersect in O, and $AO = 2OM$, $BO = 2ON$, and $CO = 2OP$. *Q.E.D.*

Note.—*The point at which the medians meet is,*

(*a*) The center of the triangle (center of area).

(*b*) The center of mass and hence called the centroid.

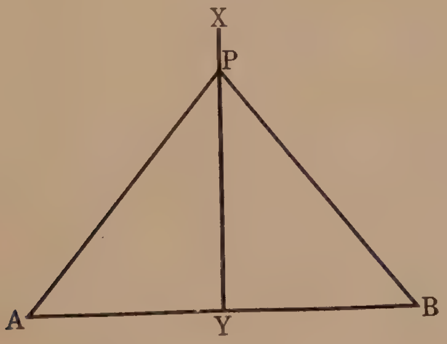
It is the point on which a triangle of uniform material will balance. It is generally called the center of gravity.

166. **Concurrent Lines.**—When three or more lines intersect in the same point, they are concurrent.

167. **The Locus of a Point.**—**Locus** means place (plural loci). The locus of a point, under a given condition, is the line or lines on which are found all the points which satisfy the given condition, but no other points. Thus, if a point moves in a plane so as to be always the same distance from a fixed point in the plane, the path will be the circumference of a circle.

To prove that a line is the locus of a point under certain conditions, we must prove, first that every point in the line satisfies the given condition and second that every point which satisfies the given condition lies in the line.

168. **Theorem.**—Every point in the perpendicular bisector of a line is equidistant from the ends of the line.



Given XY, the perpendicular bisector of AB, P any point in XY, and PA and PB, the distances from P to A and B.

To prove that $PA = PB$.

Proof.—Rt. $\triangle AYP$ = rt. $\triangle YBP$. §107.

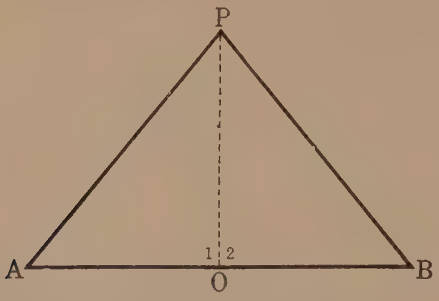
$\therefore PA = PB$. Why?

Q.E.D.

169. **Corollary** 1.—Two points each equidistant from the ends of a line determine the perpendicular bisector of the line.

170. **Corollary** 2.—The perpendicular bisectors of the sides of a triangle meet in a point. It is the center of the circumscribed circle.

171. Theorem.—Every point equidistant from the ends of a line lies in the perpendicular bisector of the line.

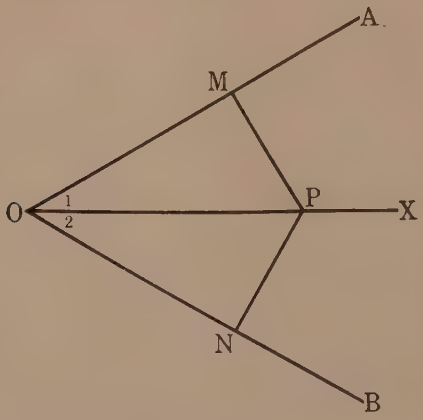


Given P, any point equidistant from A and B.
To prove that P is in the $\perp$ bisector of AB.
Proof.—Take O the mid-point of AB. Draw PO.

Then $\triangle AOP = \triangle OBP$. §111.
$\therefore \angle 1 = \angle 2$. Why?
$\therefore \angle 1$ and $\angle 2$ are rt. $\angle$s. §74.
$\therefore PO$ is the $\perp$ bisector of AB.
Q.E.D.

172. Corollary.—The perpendicular bisector of a line is the locus of all points equidistant from the ends of the line.

173. Theorem.—Every point in the bisector of an angle is equidistant from the sides of the angle, and conversely.



Given OX, the bisector of $\angle AOB$, P, any point in OX, $PM \perp OA$, and $PN \perp OB$.
To prove that $PM = PN$.
Proof.—Rt. $\triangle OPM =$ rt. $\triangle ONP$. §137.
$\therefore PM = PN$. Why?

State and prove the converse.

174. **Corollary.**—The bisectors of the angles of a triangle meet in a point. It is the center of the inscribed circle.

MISCELLANEOUS PROBLEMS

1. Write all theorems and corollaries which state that two triangles are equal.

2. Write all theorems and corollaries which state that two right triangles are equal.

3. What pairs of angles are equal when two parallel lines are cut by a transversal?

4. If two lines are cut by a transversal, what pairs of angles must be equal to make the lines parallel?

5. State the conditions under which two lines are equal; two angles are equal.

6. State the condition under which a quadrilateral is a parallelogram.

Note.—*A working knowledge of problems 1 to 6 will greatly aid in solving geometric problems.*

Prove the following:

7. The bisectors of two supplementary adjacent angles are perpendicular to each other.

8. The bisectors of two alternate interior angles of parallel lines are parallel.

9. Corresponding medians of equal triangles are equal.

10. Corresponding altitudes of equal triangles are equal.

11. Each angle of an equilateral triangle is 60°.

12. The altitudes upon the legs of an isosceles triangle are equal.

13. In an isosceles triangle, the bisector of the vertex angle bisects the base and is perpendicular to the base.

14. In an equilateral triangle the medians, the altitudes, the perpendicular bisectors of the sides, and the bisectors of the angles coincide.

15. If two sides of a triangle are extended each its own length, through a common vertex, the line joining the ends of the extended sides equals the third side of the triangle and is parallel to it. This problem is used in finding distances which cannot be measured. Illustrate.

16. In an isosceles triangle the median from the vertex to the base bisects the vertex angle and is perpendicular to the base.

17. The exterior angle at the base of an isosceles triangle is equal to one-half the vertex angle plus 90°.

18. The median to the hypotenuse of a right triangle equals one-half the hypotenuse.

19. In a right triangle if one acute angle is double the other, the hypotenuse is double the shorter leg.

20. If from any point in the base of an isosceles triangle, lines are drawn parallel to the equal sides, a parallelogram is formed whose perimeter is constant. It is equal to the sum of the two legs.

21. The bisectors of the angles of a parallelogram form a rectangle.

22. The bisectors of the angles of a rectangle form a square.

23. The base angles of an isosceles trapezoid are equal.

24. The opposite angles of an isosceles trapezoid are supplementary.

25. The diagonals of an isosceles trapezoid are equal.

26. If the diagonals of a parallelogram are perpendicular, the figure is either a square or a rhombus.

27. If a diagonal bisects an angle of a parallelogram the figure is equilateral.

28. If two altitudes of a triangle are equal, the triangle is isosceles.

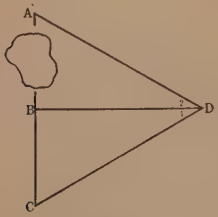
29. Lines from two opposite vertices of a parallelogram to the mid-point of the opposite sides trisect the diagonal.

30. The angle formed by the base of an isosceles triangle and the altitude upon a leg equals one-half the vertex angle.

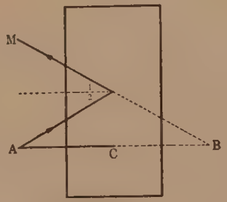
31. To find the distance AB, lay off BD any length $\perp AB$. At D lay off $\angle 1 = \angle 2$ and extend AB to meet DC at C. Why does $BC = AB$?

32. An image reflected in a mirror appears to be as far behind the mirror as the reflected object is in front of it. The object is at A but is seen at B, if the observer is at M. Show that $AC = CB$.

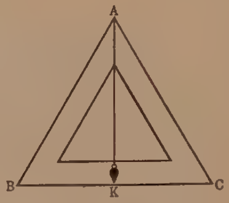
Note.—*A ray of light striking a mirror at any given angle is reflected at the same angle, i.e.,* $\angle 1 = \angle 2$.



PROB. 31.



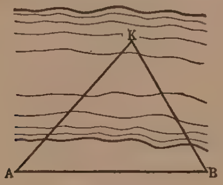
PROB. 32.



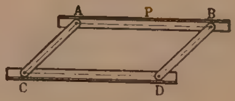
PROB. 33.

33. Using three strips, construct an isosceles triangle as shown in the figure making $AB = AC$. From the vertex suspend a plumb line. The surface supporting the triangle is level when the plumb line coincides with the mid point K of the base. Why?

34. Surveyors often locate a point such as K in the river by observations from the bank as follows: Two stakes A and B are set and the distance AB is accurately measured. Observers are stationed at A and B with transits and the angles A and B are measured while a rodman in a boat holds the point K. How may the engineers re-locate this point, providing A and B are not disturbed?



PROB. 34.



PROB. 35.

35. A parallel rule for drawing parallel lines may be constructed as shown in the figure. $CDBA$ is a parallelogram. C, D, B and A act as pivots allowing the frame to be opened or closed freely. AB and CD are straight edges. The rules are in position to draw a line through $P \parallel$ to CD.

36. Show why a floor may be laid off in regular hexagons only.

37. Why not in regular pentagons or regular octagons?

38. Why may it be laid off in a combination of regular octagons and squares?

39. The lines joining the mid-points of the sides of a triangle divide it into four equal triangles.

40. If the bisector of an exterior angle of a triangle is parallel to the base, the triangle is isosceles.

41. If the sum of two angles of a triangle is equal to the third, the triangle is a right triangle.

42. In $\triangle ABC$, if $\angle A = 73°$ and $\angle B = 68° 32'$, find $\angle C$ and the exterior angle at B.

43. If one acute angle of a right triangle is four times the other, how many degrees are there in each?

44. If a base angle of an isosceles triangle is 60°, find the vertex angle. What kind of a triangle is this?

45. If the vertex angle of an isosceles triangle equals three times the sum of the two base angles, how many degrees are there in each angle?

46. If the vertex angle of an isosceles triangle is one-fourth each of the base angles, find each angle.

47. Find the three angles formed by the bisectors of the angles of a triangle, whose angles are 54°, 68°, and 58°.

48. If $\angle A$ of $\triangle ABC$ is 35° and the exterior angle at B is 108°, find $\angle C$.

49. If the vertex angle of an isosceles triangle is one-half of either exterior angle at the extremities of the base, find each angle of the triangle. What kind of a triangle is this?

50. If one angle of a triangle is double another, the line drawn from the third vertex and making with the longer adjacent side an angle equal to the less given angle, divides the triangle into two isosceles triangles.

51. Find the number of degrees in the angle formed by the bisectors of two complementary adjacent angles; of two supplementary adjacent angles.

52. One angle of a triangle is 50°, find the number of degrees in each of the other angles if one of them is twice the other.

53. In triangle ABC, $\angle B$ is twice $\angle A$ and $\angle C$ is twice $\angle B$, find the number of degrees in each angle of the triangle.

54. If the vertex angle of an isosceles triangle is 70°, find the number of degrees in the angle formed by the bisectors of the base angles.

55. The difference between two adjacent angles of a parallelogram is 40°, find the number of degrees in each angle.

56. If one side of an equiangular pentagon is extended, how many degrees will the exterior angle contain?

57. The sum of the angles of a polygon is equal to 32 right angles, how many sides has the polygon?

58. The bisectors of the base angles of an isosceles triangle make an angle which is equal to the exterior angle at the base.

59. The angles of a triangle are $x°$, $2x°$ and $y°$, and the angle y is 8° larger than the angle x. Find the number of degrees in each angle of the triangle.

60. The angles of a triangle are $10x°$, $15x°$, and 30°. Find each angle of the triangle.

61. If the angles of a triangle are denoted by $(x + 20)°$, $(x + 16)°$, and $(10x + 24)°$, find them.

175. **Circle.**—A circle is a portion of a plane bounded by a curved line, all points of which are equally distant from a fixed point within called the **center**. This curve is the **locus** of all points in a plane at a given distance from a fixed point.

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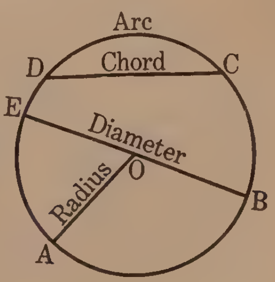


In elementary work in mathematics the bounding line is called the **circumference** of the circle. However, in more advanced work the locus itself is often regarded as the circle.

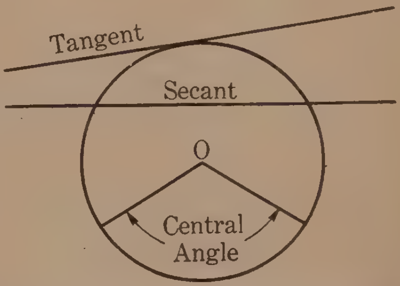
176. **Radius** (plural radii).—A radius of a circle is a line drawn from the center to any point on the circumference, as OA.

177. **Diameter.**—A diameter of a circle is a line drawn through the center and terminated by the circumference, as EB.

178. **Chord.**—A chord is a line joining any two points on the circumference of the circle, as *DC* (written $\overline{DC}$). A diameter is the greatest chord of a circle. A chord subtends an arc.

179. **Arc.**—An arc is any part of the circumference, as *DC* (written $\overset{\frown}{DC}$). Every chord divides the circumference into two arcs. If the arcs are equal each is a semi-circumference, or a semi-circle. If the arcs are unequal the smaller is called the **minor arc,** as *DC*, and the larger is called the **major arc,** as *DAC*.

180. **Secant.**—A secant is a line which cuts the circumference in two points.



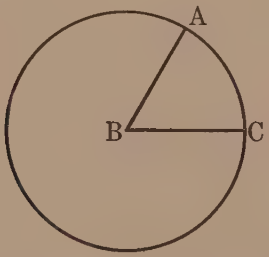
181. **Tangent.**—A tangent is a line which touches the circumference at only one point. This point is called the point of tangency, or point of contact.

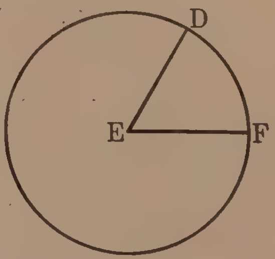
182. **Line of Centers.**—The line of centers of two circles is a line joining their centers.

183. **Central Angle.**—A central angle is an angle whose vertex is at the center and whose sides are radii of the circle. An angle, or more properly the sides of an angle, intercepts an arc, and an arc subtends an angle. Other angles will be defined as needed.

184. **Concentric Circles.**—Concentric circles are circles having the same center.

185. **Fundamental Properties of Circles.**—These properties should be remembered because of their importance in later work.





1. **Radii** of the same circle, or of equal circles are equal.
2. A **diameter** of a circle equals twice a radius.
3. In the same circle or in equal circles, equal central angles intercept equal arcs. For, if $\angle B$ is placed on $\angle E$ it will coincide. Also point *A* will coincide with *D* and *C* with *F*. §90–6 §185–1.

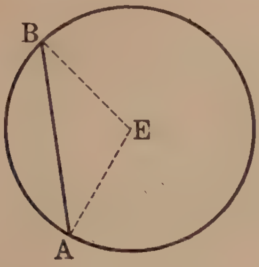
All points on the arc are equidistant from the center, therefore, $\overset{\frown}{AC}$ will coincide with $\overset{\frown}{DF}$. $\therefore \overset{\frown}{AC} = \overset{\frown}{DF}$. §90–6.

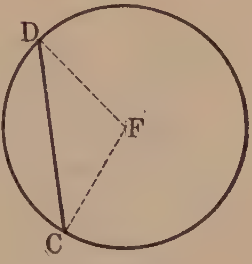
4. In the same circle or in equal circles equal arcs subtend equal angles at the center.

For if $\overset{\frown}{AC}$ is made to coincide with its equal $\overset{\frown}{DF}$, point B will coincide with point E, and therefore $\angle B$ coincides with $\angle E$.

$\therefore \angle B = \angle E$. Why?

5. In the same circle or in equal circles, equal chords subtend equal arcs.





For $\triangle AEB = \triangle CFD$. §111.

$\therefore \angle E = \angle F$ Why? $\therefore \overset{\frown}{AB} = \overset{\frown}{CD}$. §185–3.

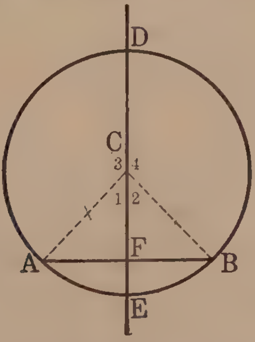
6. In the same circle or in equal circles, equal arcs are subtended by equal chords.

For $\angle E = \angle F$. §185–4.

$\therefore \triangle AEB = \triangle CFD$. Why?

$\therefore \overline{AB} = \overline{CD}$. §104.

186. **Theorem.**—A line drawn through the center of a circle perpendicular to a chord bisects the chord and the arcs which it subtends.



Given the circle C in which DE passes through C and $\perp AB$.

To prove that $AF = FB$, $\overset{\frown}{AE} = \overset{\frown}{EB}$, and $\overset{\frown}{AD} = \overset{\frown}{DB}$.

Proof.—Draw the radii CA and CB.

Then rt. $\triangle AFC =$ rt. $\triangle FBC$. Why?

$\therefore AF = FB$ and $\angle 1 = \angle 2$. Why?

$\therefore \overset{\frown}{AE} = \overset{\frown}{EB}$. §185–3.

$\angle 3 = \angle 4$. §83.

$\therefore \overset{\frown}{AD} = \overset{\frown}{BD}$. Why?

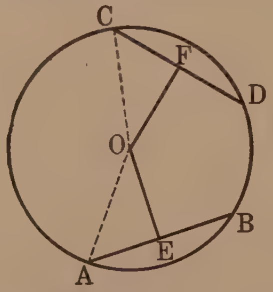
Q.E.D.

187. **Corollary 1.**—A diameter of a circle divides the circumference into two equal arcs.

188. **Corollary** 2.—A line joining the center of a circle to the midpoint of a chord is perpendicular to the chord.

189. **Corollary** 3.—The perpendicular bisector of a chord passes through the center of the circle and bisects the arcs which the chord subtends.

190. **Theorem.**—In the same circle or in equal circles, if chords are equal, they are equidistant from the center.



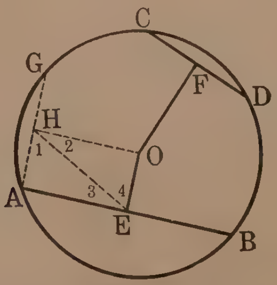
Given the circle O in which $\overline{AB} = \overline{CD}$, $OE \perp AB$, and $OF \perp CD$.
To prove that $OE = OF$.
Proof.—Draw the radii OA and OC.

OE bisects AB and OF bisects CD. §186.
Then rt. $\triangle AEO =$ rt. $\triangle OFC$. §142.
$\therefore OE = OF$. Why?
Q.E.D.

191. **Theorem.**—In the same circle or in equal circles, if chords are equidistant from the center they are equal.

Use the figure of §190.
Given $OE = OF$.
To prove that $\overline{AB} = \overline{CD}$.
Proof. Rt. $\triangle AEO =$ rt. $\triangle OFC$. Why?
$\therefore AE = CF$. Why?
$\therefore 2\,AE = 2\,CF$, or $\overline{AB} = \overline{CD}$. Why?
Q.E.D.

192. **Theorem.**—In the same circle or in equal circles, if two chords are unequal, the greater chord is nearer the center.



Given the circle O in which $\overline{AB} > \overline{CD}$, $OE \perp AB$, and $OF \perp CD$.
To prove that $OE < OF$.

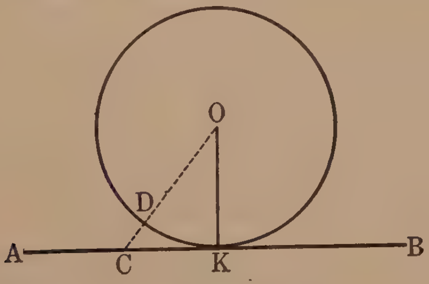
Proof.—Draw $\overline{AG} = \overline{CD}$ and $OH \perp \overline{AG}$. Draw HE.

Then $OH = OF$, §190.
$AE = \frac{1}{2} AB$, and $AH = \frac{1}{2} AG$. §186.
$\therefore AE > AH$. *Ax.* 11.
In $\triangle AEH$, $\angle 1 > \angle 3$. §143.
$\angle AHO = \angle AEO$. Why?
$\therefore \angle 2 < \angle 4$. *Ax.* 10.
$\therefore$ In $\triangle HEO$, $OH > OE$. §144.
$\therefore OF > OE$. *Ax.* 5.
Q.E.D.

193. **Theorem.**—In the same circle or in equal circles, if two chords are unequally distant from the center, the one nearer the center is the greater.

Use the figure of §192.
Given $OF > OE$.
To prove that $AB > CD$.
Proof.—$OF > OE$. Given.
$\therefore OH > OE$. *Ax.* 5.
In $\triangle HEO$, $\angle 4 > 2$. Why?
In $\triangle AEH$, $\angle 3 < 1$. Why?
$\therefore AE > AH$ and $2AE > 2AH$. i.e. $AB > AG$. Why?
$\therefore \overline{AB} > \overline{CD}$. Why?
Q.E.D.

194. **Theorem.**—A line tangent to a circle is perpendicular to the radius drawn to the point of tangency.



Given AB tangent to the circle O at point K.
To prove that $OK \perp AB$.
Proof.—Take C any point on AB except K and draw OC.
Then the point C is outside the circle O. §181.
Hence $OC > OD$ or its equal OK. *Ax.* 13.
$\therefore OK \perp AB$; *i.e.*, $AB \perp OK$. §145.
Q.E.D.

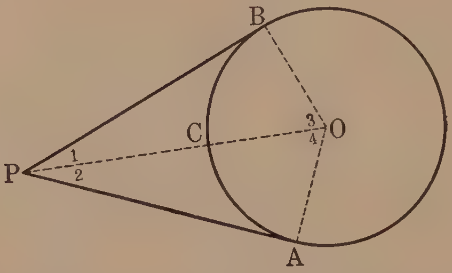
195. **Corollary** 1.—A line perpendicular to a radius at its outer extremity is tangent to the circle.

196. **Corollary** 2.—A line perpendicular to a tangent at the point of tangency passes through the center of the circle.

197. **Corollary** 3.—A diameter perpendicular to a tangent passes through the point of tangency.

198. **Corollary** 4.—Tangents to a circle at the extremities of a diameter are parallel.

199. **Theorem.**—Two tangents drawn to a circle from a given external point are equal.



Given PA and PB tangents to the circle O from a given point P.

To prove that $PA = PB$.

Proof.—Draw the radii OA and OB, and the line PO.

Then $OA \perp PA$ and $OB \perp PB$. §194.

Rt. $\triangle PAO =$ rt. $\triangle POB$. §142.

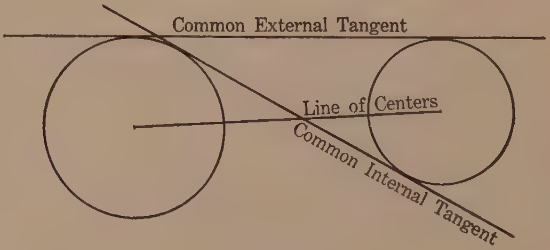
$\therefore PA = PB$. Why?

Q.E.D.

Show that $\angle 1 = \angle 2$; $\angle 3 = \angle 4$; $\widehat{BC} = \widehat{CA}$; $PO \perp$ bisector of $\overline{AB}$.

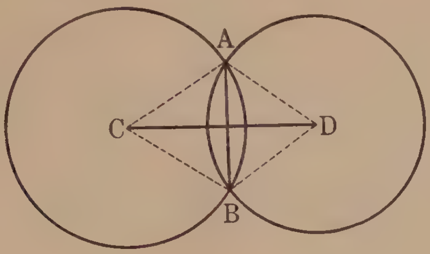
200. **Common Tangents.**—A common tangent to two circles is a line tangent to both circles. A common tangent which intersects the line of centers is a **common internal tangent.**

A common tangent which does not intersect the line of centers is a **common external tangent.** How many such tangents may be drawn to two circles?



201. **Tangent Circles.**—Two circles tangent to the same line at the same point are tangent circles. Two circles are tangent **internally** when they lie on the same side of the common tangent, and tangent **externally** when they lie on opposite sides of the common tangent. Illustrate by use of drawings.

202. **Theorem.**—If two circles intersect, the line of centers is the perpendicular bisector of their common chord.



Given circles *C* and *D* intersecting at points *A* and *B*, *CD* the line of centers and *AB* the common chord.

To prove that *CD* is the ⊥ bisector of *AB*.

Proof.—Draw the radii *CA*, *CB*, *DA*, and *DB*.

C and *D* are each equidistant from *A* and *B*. Why?

∴ *CD* is the perpendicular bisector of *AB*. §169.

Q.E.D.

CONSTRUCTIONS

203. **Problems of Construction.**—In the work thus far it has been assumed that the figures needed for proving theorems and solving problems could be drawn accurately enough to serve their purpose. Now figures will be constructed by methods which can be proven to be as exact as it is possible to construct them by the use of the **straight edge and compasses,** the only instruments allowed in geometric construction.

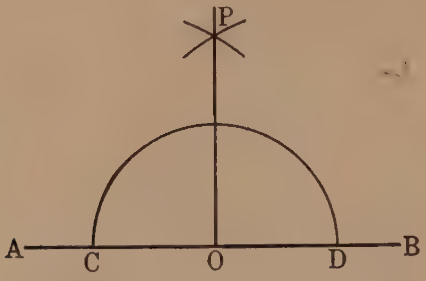
204. **Assumptions.**—It is necessary to agree that a few simple operations which underlie all construction work are possible. Thus:

1. Through any two points a straight line can be drawn.
2. A straight line may be extended indefinitely or it may be limited by any two points.
3. A circle can be described with any given point as a center, and any given line as a radius.

205. **Solution of a Construction Problem.**—The solution of a problem is naturally divided as follows:

1. **Analysis.**—It must be determined from the statement of the problem what is given and what is required.
2. **The Construction.**—It will often be a great help in making a construction if a trial sketch of the required figure is made before using the straight edge and compasses.
3. **Proof.**—After the required figure is constructed, the theorems of geometry will be used in proving that the figure is the one required. All given and required lines are drawn full lines. Auxiliary lines are used to aid in the proof.

206. **Problem.**—To erect a perpendicular to a given line at a given point on the line.



Given the line AB and point O on AB.

To construct a $\perp$ to AB at O.

Construction.—With O as a center describe an arc meeting AB at C and D. With C and D as centers and a radius greater than OD describe arcs intersecting at P. Draw PO.

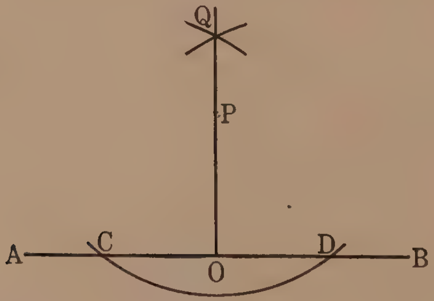
Then PO is the perpendicular required.

Proof.—P and O are each equidistant from C and D by construction, hence determine the perpendicular bisector of CD.

$\therefore PO$ is $\perp AB$. §169.

Q.E.F.

207. **Problem.**—To draw a perpendicular from a given point to a given line.



Given the point P and the line AB.

To construct a $\perp$ from P to AB.

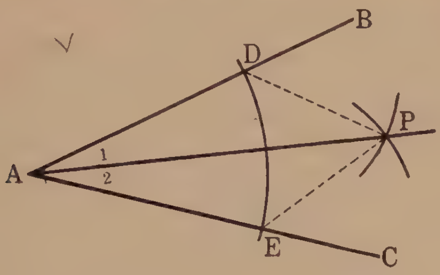
Construction.—With P as a center and any convenient radius describe an arc intersecting AB in C and D. With C and D as centers and a radius greater than one-half CD, describe arcs intersecting at Q. Draw PQ and extend it to meet AB in O.

PO is the perpendicular required.

Proof.—P and Q are each equidistant from C and D by construction. Hence P and Q determine the perpendicular bisector of CD.

$\therefore PO \perp AB$. *Q.E.F.*

208. **Problem.**—To bisect a given angle.



Given the $\angle A$.

To bisect $\angle A$.

Construction.—With A as a center and any convenient radius describe an arc intersecting AB in D and AC in E.

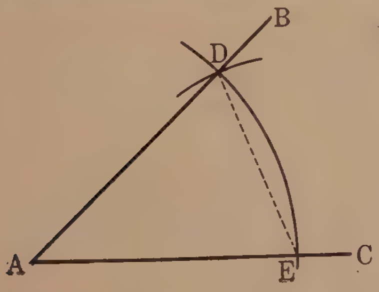
With D and E as centers and a radius greater than one-half DE, describe arcs intersecting at P. Draw AP.

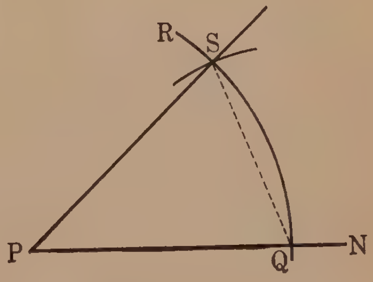
Then AP is the bisector required.

Proof.—Draw PD and PE.

Then $\triangle AEP = \triangle APD$. §111.
$\therefore \angle 1 = \angle 2$. §104.
$\therefore \angle A$ is bisected by AP. Q.E.F.

209. **Problem.**—To construct an angle equal to a given angle, given one side and the vertex.





Given $\angle BAC$ and vertex P on the side PN.

To construct at P an angle equal to $\angle A$.

Construction.—With A as a center and any convenient radius describe an arc intersecting AB in D and AC in E.

With P as a center and with the same radius describe an arc RQ intersecting PN at Q. With Q as a center and a radius equal to DE, describe an arc intersecting arc RQ in S. Draw PS.

Then $\angle P$ is the angle required.

Proof.—Draw DE and QS.

Then $\triangle AED = \triangle PQS$. Why?
$\therefore \angle A = \angle P$. Why?

Problems

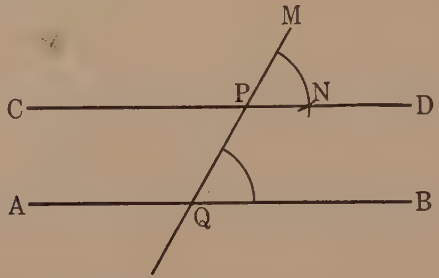
1. Construct an equilateral triangle.

2. Construct the following angles: 90°, 45°, 60°, 30°, 15°, 75°, 105°, 120°, 135°, 150°, 7.5°, 22°30′.

3. Trisect a right ∠.

4. Can any given angle be trisected by using only the straight edge and compasses?

210. **Problem.**—To construct a line parallel to a given line through a given external point.



Given line AB and point P.

To construct a line ∥ AB through P.

Construction.—Through P draw any line intersecting AB ~~at~~ Q.

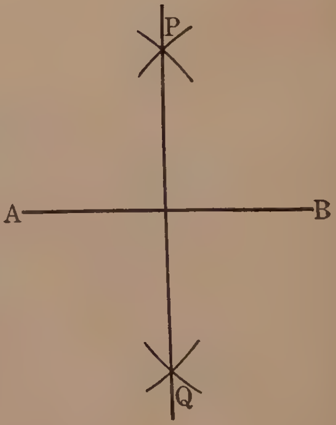
At P construct an $\angle MPN$ equal to the $\angle MQB$, as in the above figure. §209.

Draw CD through P and N.

Then CD is the line required.

Proof.—If two straight lines are cut by a third, making the corresponding angles equal the lines are parallel.

Construct CD parallel to AB through P using §§123 and 125.



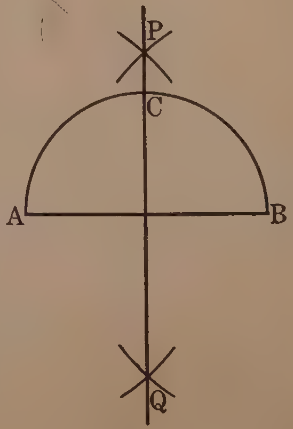
211. **Problem.**—To bisect a given straight line.

Given line AB.

To bisect AB.

Construction.—With A and B as centers and a radius greater than one-half AB describe arcs intersecting at P and Q. Draw PQ.

Proof.—PQ is the ⊥ bisector of AB. §169.



212. **Problem.**—To bisect a given arc.

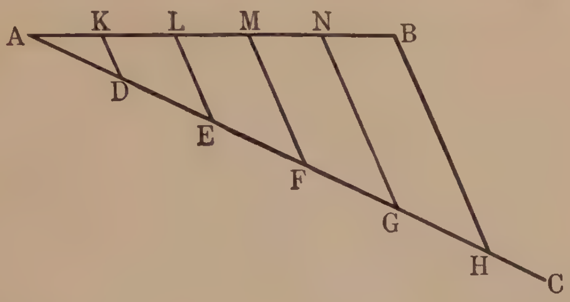
Given arc AB.

To bisect arc AB.

Construction.—Draw the chord AB and construct PQ the perpendicular bisector of $\overline{AB}$. §211.

Proof.—PQ bisect $\overset{\frown}{AB}$ at C. §189.

213. **Problem.**—To divide a given line into any number of equal parts.



Given the line AB.

To divide AB into any number of equal parts. (For example five parts.)

Construction.—Draw AC making any convenient angle with AB. With the compasses mark off on AC five equal distances from A. From H the fifth point of division draw HB. Through D, E, F, and G construct lines $\parallel$ HB meeting AB in K, L, M, and N.

Proof.—The divisions AK, KL, etc., are the divisions required. §162.

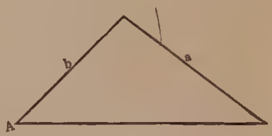
CONSTRUCTION PROBLEMS

In every problem prove that the construction is correct.

1. Divide a given line into 3 equal parts.
2. Divide a given line into 12 equal parts.
3. Construct a triangle having given.
   - (a) Two sides and the included angle.
   - (b) One side and two adjoining angles.
   - (c) Three sides.
4. Construct a triangle having given two sides and an angle opposite one of them.

Given A, a, and b.

Name the cases in which a triangle is impossible.



(a) $\angle A$ acute, $a = b$. (b) $\angle A$ acute, $a > b$.
(c) $\angle A$ acute, $a < b$. (d) $\angle A$ obtuse, $a = b$.
(e) $\angle A$ obtuse, $a < b$. (f) $\angle A$ obtuse, $a > b$.
(g) $\angle A$ a rt. $\angle$, $a = b$. (h) $\angle A$ a rt. $\angle$, $a < b$.
(i) $\angle A$ a rt. $\angle$, $a > b$.

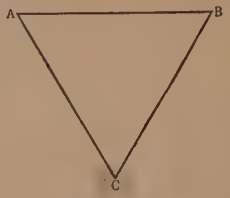
5. Construct a right triangle having given:
   - (a) A leg and an acute angle.
   - (b) The hypotenuse and an acute angle.
   - (c) The hypotenuse and a leg.
6. Circumscribe a circle about a given triangle. §168.
7. Construct a circle through any three points not in the same straight line.
8. Inscribe a circle in a given triangle. §174.
9. Construct a tangent to a circle at a point on the circumference.
10. Given the perimeter, construct an equilateral triangle.
11. Given the altitude and a base angle, construct an isosceles triangle.

12. Given the perimeter and the base, construct an isosceles triangle.

13. Given the diagonal, construct a square.

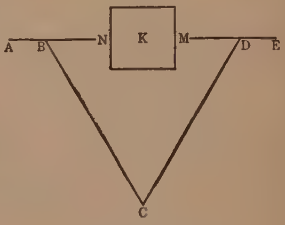
14. Construct a circle passing through two points, with its center on a given line.

15. To lay off a 60° angle using a 100 ft. metallic tape. Lay off a straight line AB 50 ft. long. Fasten the zero end of the tape at A, the 100 ft. end at B, and stretch the tape as shown in the figure, locating C at the 50 ft. mark. Why is each angle 60°? Construct to scale.



Prob. 15.

16. To continue a straight line through an obstacle such as a building. AB is to be extended beyond the building K. Extend AB to N. At B lay off a 60° angle NBC and run the line BC to a point C clearing the building and allowing space to complete the triangle. At C lay off a 60° angle and run $CD = BC$. At D lay off a 60° angle CDM locating the line ME. Why is ME a continuation of the line AB? Construct to scale.



Prob. 16.

LOCUS PROBLEMS

Read again §167.

Draw the following loci, all points being in the same plane.

1. A point 2 inches from a given point.

2. A point 2 inches from a given line.

3. A point equidistant from two intersecting lines.

4. What is the value of the angle between the two lines representing the locus in problem 3?

5. A point equidistant from two intersecting lines and 3 inches from the point of intersection.

6. A point equidistant from two parallel lines.

7. A point equidistant from three given points not in a straight line. Why is this construction impossible if the three points are on the same straight line?

8. Surveyors are often required to find the locus of a point whose marks of identification have been lost. Thus, locate a point 13.5 ft. from A and 22.6 ft. from B. The location of points A and B are accurately known.

9. Locate a point described as being 15 ft. from a certain point, A, and 18 ft. from a given line. Hint. *Find first the locus of all points 15 ft. from the given point. Second find the locus of all points 18 ft. from the given line. The locus common to both of these will at once suggest itself.*

10. Find the locus of a point within a triangle equidistant from the three sides.

11. Find a point or points equidistant from two || lines and at a given distance from a given point. What conditions will result in the finding of one point only; two points. Show why this construction may be impossible.

12. Find the locus of points equidistant from two given points and equidistant from two || lines. Discuss fully.

CHAPTER V

PRODUCTS AND FACTORS

214. **Product.**—When two or more expressions are multiplied together the result is a product.

215. **Factors.**—The expressions multiplied together to form a product are the factors of that product.

Illustrations.—1. $2 \cdot 2 \cdot 3 = 12$. 2, 2, and 3 are the factors of 12.

2. $a(b + c) = ab + ac$. a and $(b + c)$ are the factors of the product $ab + ac$. ab is also a product for it means a times b. Therefore a and b are the factors of ab.

3. $(a + 2)(a + 3) = a^2 + 5a + 6$. The quantities $(a + 2)$ and $(a + 3)$ are the factors of $a^2 + 5a + 6$.

From the above illustrations it is evident that we must be able to multiply numbers and quantities together before we can factor any expression.

216. **Prime Factors.**—A prime factor is a number which has no factors except itself and unity. Thus 2, 2, and 3 are the prime factors of 12. a and $(b + c)$ are the prime factors of $ab + ac$. When a number or a quantity is to be factored, it is understood that the prime factors are required.

217. **Factoring.**—Factoring is the process of resolving an expression into its factors. It is the reverse of multiplication. In the study of arithmetic the multiplication tables were memorized as a foundation for accurate and rapid computation in later work. In algebra the same kind of preparation is necessary for handling the operations of multiplication and division by inspection, and in fact wherever factoring is required. We know the factors of an expression only because we know the expressions which when multiplied together will produce the expression to be factored.

218. **The Product** of a **Monomial and a Polynomial.**

Study again §41.

To factor an expression containing a monomial factor, (*a*) *Determine by inspection the monomial factor common to all the terms of the given expression.* (*b*) *The other factor is obtained by dividing the given expression by the monomial factor.*

Illustrations.—1. $mx + my = m(x + y)$.

2. $mx - my = m(x - y)$.

3. $3a^3 - 6a^2b = 3a^2(a - 2b)$.

4. $3x - 3x^2 + 3x^3 = 3x(1 - x + x^2)$.

PROBLEMS

Find the product of the following:

1. $b(a + c)$.
2. $x(m - n)$.
3. $k(l + w)$.
4. $d(a - b + c)$.
5. $s(r - k + v - y + c)$.
6. $p(t - u + x - k)$.

Factor the following and note carefully the six preceding problems.

7. $ab + bc$.
8. $xm - xn$.
9. $kl + kw$.
10. $da - db + dc$.
11. $er - ek + ev - ey + ec$.
12. $pt - pn + px - pk$.

Remove the parentheses in the following:

13. $3a(a - 2)$.
14. $5k(k + 2)$.
15. $2b(b - 2)$.
16. $-3d(d - 2)$.
17. $5y(y + 3)$.
18. $a^2(a + 2ab + b^2)$.
19. $3m(1 + n)$.
20. $3a(a^2 - ab - 2b^2)$.
21. $2r(r - 1)$.
22. $2x(x^2 - 3x + 2)$.
23. $3d(2d^2 - d + 1)$.
24. $m^2n(n^2 + 2m - 4)$.
25. $am(a^2 - a + 1)$.
26. $x^2y(1 + 2x - x^2 + 2x^3)$.

Factor the following.

27. $3a^2 + 6a$.
28. $3m - 3mn$.
29. $ab + ab^2$.
30. $2r + r^2$.
31. $5y^2 - 15y$.
32. $3r + 5r^3$.
33. $6d^3 + 3d - 2d^2$.
34. $10k - 5k^2$.
35. $a^3m + a^2m + am$.
36. $3d^2 - 6d$.
37. $3a^3 - 6a^2b + 3ab^2$.
38. $a^2 + 2a^2b + ab^2$.
39. $2x^3 - 8x^2 + 8x$.
40. $m^2n^3 - m^3n + m^2n$.
41. $xy^2 - 2x^2y^2 + x^3y^2 + x^4y^2$.
42. $m^2(a + b) + m(a + b)$.
43. $m^2p + mp^2 + mp$.
44. $y^2(x - 1) - y^2x(x - 1)$.

219. The Product of Two Binomials Having a Common Term.—The multiplication of terms and quantities must now be practiced so that the products may be found quickly without actually writing all the steps of the work.

Thus, the method now is,

$$(a + 2)(a + 3) = a^2 + 5a + 6.$$

Instead of

$$\begin{array}{r} a + 2 \\ a + 3 \\ \hline a^2 + 2a \qquad \\ + 3a + 6 \\ \hline a^2 + 5a + 6. \end{array}$$

10c Math

§42.

The first is shorter and therefore saves time.

Notice the steps $(a)(a) + [(3)(a) + (2)(a)] + (2)(3)$,

or $a^2 + 5a + 6$.

This abbreviated method may be expressed as follows:

The product of two binomials having a common term is equal to the square of the common term, plus the algebraic sum of the products of the common term and the unlike terms, plus the product of the unlike terms.

It follows that the factors of an expression of the type $x^2 + bx + c$ are two binomials. One term of each factor must be **x** *and the algebraic sum of the unlike terms must be* **b** *and their product* **c**.

PROBLEMS

Find the product of the following:

1. $(a + 2)(a + 4)$.
2. $(b - 7)(b + 11)$.
3. $(x - 2)(x - 3)$.
4. $(e + 9)(e - 7)$.
5. $(m + 5)(m - 2)$.
6. $(s - 13)(s + 7)$.
7. $(m + 2)(m + 2)$.
8. $(p + 14)(p - 7)$.
9. $(r - 3)(r - 3)$.
10. $(m - 13)(m + 6)$.
11. $(p - 6)(p + 2)$.
12. $(m - 13)(m - 6)$.
13. $(x + 9)(x + 6)$.
14. $(m + 13)(m + 6)$.
15. $(r - 9)(r + 11)$.
16. $(z + 9)(z + 13)$.
17. $(a + 4)(a - 7)$.
18. $(n - 12)(n + 13)$.

Factor the following:

19. $a^2 - 6a + 8$.
20. $e^2 - 2e - 63$.
21. $x^2 - x - 6$.
22. $s^2 - 6s - 91$.
23. $m^2 + 3m - 10$.
24. $p^2 - 7p - 98$.
25. $m^2 - 4m + 4$.
26. $m^2 - 7m - 78$.
27. $r^2 + 6r + 9$.
28. $m^2 + 19m + 78$.
29. $p^2 - 4p - 12$.
30. $m^2 - 19m + 78$.
31. $x^2 - 15x + 54$.
32. $r^2 - 2r - 99$.
33. $b^2 - 4b - 77$.

220. The Product of Binomials of the Type $(mx + p)(nx + r)$.

By multiplication, $(2x - 3)(3x + 2) = 6x^2 - 5x - 6$.

That is, $(2x)(3x) + [2(2x) - 3(3x)] + (-3)(2)$,

or $6x^2 - 5x - 6$.

Hence, the product of two binomials having all terms unlike is equal to the product of the first terms of the binomials, plus the algebraic sum of the cross products, plus the product of the second terms. The product is readily obtained as shown by the following arrangement.

$$\begin{aligned}(2x - 3)(3x + 2) &= 6x^2 + (4x - 9x) - 6.\\ &= 6x^2 - 5x - 6.\end{aligned}$$

It follows that the factors of an expression of the type $ax^2 + bx + c$ are two binomials. The product of two of the terms of the factors is ax^2 and the product of the other two terms is c, so arranged that the algebraic sum of their cross-products is bx.

The above type can be factored when, and only when, $b^2 - 4ac$ is a square. The proof of this is found in §400.

PROBLEMS

Find the following products:

1. $(2x + 3)(x - 1)$.
2. $(3x - 2)(2x + 3)$.
3. $(x + 2)(3x + 5)$.
4. $(3b - 1)(b + 2)$.
5. $(2x - 3)(x - 2)$.
6. $(3b + 1)(b + 2)$.
7. $(2x + 3)(x + 2)$.
8. $(3b - 1)(b - 2)$.
9. $(2x - 3)(x + 2)$.
10. $(2ax - 3)(4ax + 1)$.
11. $(5a - 2)(2a - 1)$.
12. $(2rs + 5)(3rs - 4)$.
13. $(7ab + 2)(8ab - 5)$.
14. $(12cd + 9)(2cd - 3)$.
15. $(5r^2p + 9)(3r^2p + 7)$.

Factor the following, when possible.

16. $2x^2 + 5x + 3$.
17. $3b^2 + 7b + 2$.
18. $3x^2 + 11x + 10$.
19. $3b^2 - 7b + 2$.
20. $2x^2 - 7x + 6$.
21. $12a^2x^2 - 9ax - 3$.
22. $2x^2 + x + 6$.
23. $6r^2s^2 + 7rs - 20$.
24. $2x^2 + 7x + 6$.
25. $56a^2b^2 + 51ab + 10$.
26. $10a^2 - a + 2$.
27. $20c^2d^2 + 12cd - 27$.
28. $6x^2 + 5x - 6$.
29. $56p^2q^2 - 113pq + 56$.
30. $3b^2 + 5b - 2$.

221. **Square of Binomials of the Types** $(x + y)$ **and** $(x - y)$. The first type is called the square of the sum of two numbers and the second type, the square of the difference of two numbers.

That is, $(x + y)(x + y) = (x + y)^2 = x^2 + 2xy + y^2$,
and $(x - y)(x - y) = (x - y)^2 = x^2 - 2xy + y^2$.

Hence, the square of the sum of two numbers is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.

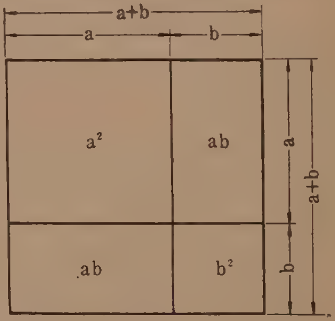
The square of the difference of two numbers is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.

These types may be combined as follows:

$$(x \pm y)^2 = x^2 \pm 2xy + y^2.$$

Note the use of the double sign (±). Thus, $\pm 2xy$ is read plus or minus $2xy$.

Illustrations. 1. Study carefully the following drawing and the areas and dimensions of each of its divisions. It proves graphically that the square constructed upon the line $(a + b)$ is equal to the sum of the squares $\mathbf{a}^2$ and $\mathbf{b}^2$ and two rectangles ($\mathbf{a}$ by $\mathbf{b}$).



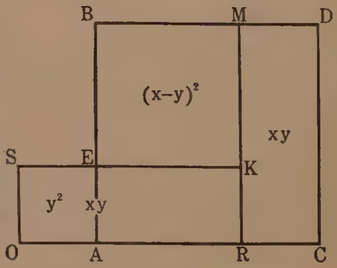
Therefore $(a + b)^2 = a^2 + b^2 + 2ab$.

2. Again study the graphic proof of the square of the difference of two numbers, as $(x - y)^2 = x^2 - 2xy + y^2$.

Let $AB = x$, $AE = y$, and $EB = x - y$.

Then it is evident that $EKMB = ACDB + AESO - (ORKS + RCDM)$ or $(x - y)^2 = x^2 + y^2 - 2xy$.

Hence, the factors of a trinomial square are two equal binomials. The terms of the binomial are the positive square roots of the squares. These terms are separated by the sign of the remaining term of the trinomial.



BROBLEMS

Find the values of the following:

1. $(x + a)(x + a)$. 2. $(2x - 1)^2$. 3. $(x + a)^2$.
4. $(3x - 2)(3x - 2)$. 5. $(x - b)(x - b)$. 6. $(2x - 5)^2$.
7. $(x + 7)^2$. 8. $(3x - 2b)^2$. 9. $(x + m)(x + m)$.
10. $(2r - 3s)^2$. 11. $(x - 3)(x - 3)$. 12. $(3x + a)^2$.
13. $(x + 6)^2$. 14. $(4a - 3q)^2$. 15. $(x - 6)^2$.
16. $(6m - 7s)^2$. 17. $(x + 10)^2$. 18. $(10x - 7y)^2$.
19. $(x - 9)^2$. 20. $(12m^2 + 9b)^2$. 21. $(3r - 4s)^2$.

Factor the following:

22. $x^2 + 2xy + y^2$.
23. $x^2 - 12x + 36$.
24. $x^2 - 2xy + y^2$.
25. $x^2 + 18x + 81$.
26. $x^2 - 4x + 4$.
27. $x^2 - 20x + 100$.
28. $x^2 + 2xa + a^2$.
29. $4x^2 + 4x + 1$.
30. $x^2 + 2bx + b^2$.
31. $9x^2 + 12x + 4$.
32. $x^2 + 6x + 9$.
33. $4x^2 - 12x + 9$.
34. $x^2 + 10x + 25$.
35. $9a^2 - 12ab + 4b^2$.
36. $4x^2 - 12xy + 9y^2$.
37. $9x^2 + 6xy + y^2$.
38. $36r^2 + 84rs + 49s^2$.
39. $16a^2 - 24ak + 9k^2$.
40. $100x^2 - 140xy^2 + 49y^4$.
41. $144a^4 + 216a^2b + 81b^2$.
42. $121x^2 - 66xy + 9y^2$.

222. **Application.**—The methods in §221 are very useful for squaring arithmetic numbers.

Illustrations. 1. $(14)^2 = (10 + 4)^2 = 100 + 80 + 16 = 196$.

2. $(18)^2 = (20 - 2)^2 = 400 - 80 + 4 = 324$.

PROBLEMS

Square the following numbers:
13, 19, 82, 36, 17, 25, 16, 73, 123, 112, 157, 128.

223. **The Product of the Sum and Difference of Two Numbers.** By multiplication, $(a + b)(a - b) = a^2 - b^2$ and $(3x + 2y)(3x - 2y) = 9x^2 - 4y^2$.

Hence, the product of the sum and difference of two numbers is equal to the square of the first minus the square of the second.

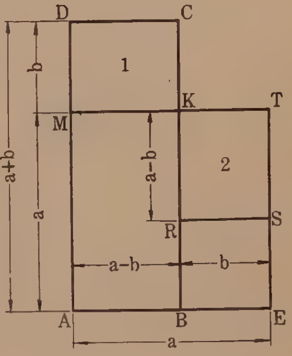


Illustration. $AE = a \therefore AETM = a^2$.
$BE = b \therefore BESR = b^2$.
$AETM - BESR = ABRSTM$.
$a^2 - b^2 = ABRSTM$.
But $ABCD = ABRSTM$, since
$(1) = (2)$.
Again, $ABCD = (a - b)(a + b)$.
$\therefore a^2 - b^2 = (a - b)(a + b)$ by substitution.

Hence, the factors of the difference of two squares are two binomials. One factor is the sum and the other factor is the difference of the positive square roots of the terms. The terms of the factors must be the same as in the given expression.

Thus, $x^2 - y^2 = (x + y)(x - y)$ not $(x + y)(y - x)$.

PROBLEMS

Find the following products:

1. $(p + x)(p - x)$.
2. $(a + m)(a - m)$.
3. $(m + 3)(m - 3)$.
4. $(1 - x)(1 + x)$.
5. $(x - 7)(x + 7)$.
6. $(x - 11(x + 11)$.
7. $(2a - d)(2a + d)$.
8. $(5ab + 3)(5ab - 3)$.
9. $(5ab - x)(5ab + x)$.
10. $(3m - 5y)(3m + 5y)$.
11. $x(x - 1)(x + 1)$.
12. $x^3(x - y)(x + y)$.
13. $a(a + b)(a - b)$.
14. $5(3x - 5)(3x + 5)$.
15. $y(x - \frac{1}{2})(x + \frac{1}{2})$.
16. $b^2(b + \frac{2}{3})(b - \frac{2}{3})$.
17. $3(2r - \frac{1}{2})(2r + \frac{1}{2})$.
18. $a(mn - p^2)(mn + p^2)$.

Factor the following.

19. $p^2 - q^2$.
20. $a^2 - b^2$.
21. $m^2 - 1$.
22. $1 - a^2$.
23. $x^2 - 4$.
24. $4a^2 - b^2$.
25. $25a^2b^2 - 9$.
26. $25a^2b^2 - c^2$.
27. $9x^2 - 25y^2$.
28. $x(x^2 - 1)$.
29. $xy(x^2 - y^2)$.
30. $a^3 - ab^2$.
31. $28x^2 - 63$.
32. $x^2 - \frac{1}{4}$.
33. $b^2 - \frac{4}{9}$.
34. $y - y^5$.
35. $\frac{4}{9}x^2 - \frac{16}{25}y^2$.
36. $\frac{1}{4}m^2 - \frac{1}{9}n^2$.

224. Special Cases of the Type $x^2 - y^2$.

Illustrations.

1. By abbreviated multiplication,

$$\begin{aligned}(a + b + c)(a - b + c) &= [(a + c) + b][(a + c) - b]\\ &= (a + c)^2 - b^2\\ &= a^2 + 2ac + c^2 - b^2.\end{aligned}$$

2. Factor $x^2 - y^2 - 6y - 9$.

Solution.—Enclosing the last three terms in parentheses,

$$\begin{aligned}x^2 - y^2 - 6y - 9 &= x^2 - (y^2 + 6y + 9)\\ &= x^2 - (y + 3)^2\\ &= [x + (y + 3)][x - (y + 3)]\\ &= (x + y + 3)(x - y - 3).\end{aligned}$$

3. By abbreviated multiplication,

$$\begin{aligned}(x^2 + 3xy + y^2)(x^2 - 3xy + y^2) &= [(x^2 + y^2) + 3xy][(x^2 + y^2) - 3xy]\\ &= (x^2 + y^2)^2 - (3xy)^2\\ &= x^4 + 2x^2y^2 + y^4 - 9x^2y^2\\ &= x^4 - 7x^2y^2 + y^4.\end{aligned}$$

4. Factor $a^4 - 23a^2b^2 + b^4$.

Solution.—Add and subtract $25a^2b^2$, which is a square, and we have $a^4 + 2a^2b^2 + b^4 - 25a^2b^2$.

$$\begin{aligned}\text{Now } a^4 + 2a^2b^2 + b^4 - 25a^2b^2 &= (a^2 + b^2)^2 - (5ab)^2\\ &= (a^2 + b^2 + 5ab)(a^2 + b^2 - 5ab)\\ &= (a^2 + 5ab + b^2)(a^2 - 5ab + b^2).\end{aligned}$$

Hence to factor an expression of the type $x^4 + bx^2y^2 + y^4$, add and subtract the same square in order to get an expression in the form of the difference of two squares.

PROBLEMS

Factor the following:

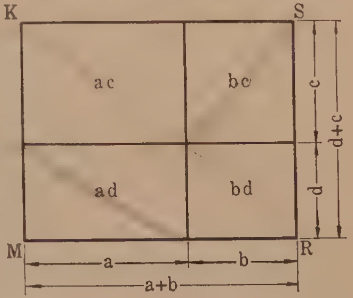
1. $a^2 + 4ab + 4b^2 - x^2$.
2. $a^2 - 2ab + b^2 - c^2$.
3. $9x^2 - a^2 + 4ab - 4b^2$.
4. $a^2 - 2xy - x^2 - y^2$.
5. $a^2 - b^2 - 1 - 2b$.
6. $4m^2 + 6n - n^2 - 9$.
7. $10bx + 25b^2 + x^2 - 100$.
8. $1 - 2xy - x^2 - y^2$.
9. $9a^2 - 6a + 1 - 25b^2 + 20bx - 4x^2$.
10. $m^2 + 12mn + 36n^2 - 10y - 25y^2 - 1$.
11. $a^4 + a^2 + 1$.
12. $4x^4 - 13x^2y^2 + y^4$.
13. $9m^4 + 3m^2n^2 + 4n^4$.
14. $16y^4 - 9y^2 + 1$.
15. $x^4 + x^2y^2 + y^4$.
16. $9a^4 + 3a^2b^2 + 4b^4$.
17. $100n^4 - 61n^2 + 9$.
18. $a^4 + 4b^4$.
19. $1 + 64x^4$.
20. $32c^4 + 2d^4 - 56c^2d^2$.

225. The Product of Binomials of the Type $(a + b)(c + d)$.

By multiplication, $(a + b)(c + d) = ac + bc + ad + bd$.

Illustrations.

1. Consider $(a + b)(c + d) = ac + ad + bc + bd$ shown graphically. The figure is composed of four rectangles with dimensions as indicated.



The rectangle $MRSK = (a + b)(c + d)$,
and also $MRSK = ac + bc + bd + ad$.

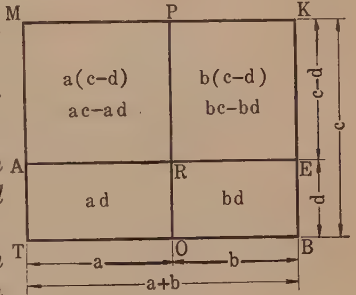
2. Again, $(a + b)(c - d) = ac - ad + bc - bd$.

The rectangle $AEKM = (c - d)(a + b)$.

But $AEKM = ARPM + REKP$.

Also $AEKM = a(c - d) + b(c - d)$.

$\therefore AEKM = ac - ad + bc - bd$.



Hence, to factor $ac + bc + ad + bd$,

(a) *Group terms having a common monomial factor. Thus,* $(ac + bc) + (ad + bd)$.

(b) *Take out the monomial factor in each group which will give the same expression in the parentheses. Thus,* $c(a + b) + d(a + b)$.

(c) *Add the coefficients of the common expression* $(a + b)$ *and get* $(c + d)(a + b)$.

Note.—*The addition of coefficients in the above problem is the same as in* $3(a + b) + 5(a + b) = (3 + 5)(a + b)$ or $8(a + b)$.

PROBLEMS

Find the products:

1. $(a + b)(c + d)$.
3. $(a + 2)(c - 3)$.
5. $(x - 1)(x + 2)(x + 1)$.
7. $(ax + 2)(x - 1)$.
9. $(x - 1)(a - b - c)$.

2. $(a - c)(b - d)$.
4. $(2 - x)(5 + y)$.
6. $(a - b)(a + b)(a - b)$.
8. $(m^2 + p)(m - n)$.
10. $(c + d - b)(cd - a)$.

Find the factors:

11. $ac + bc + ad + bd$.
13. $ab + ad - bc - cd$.
15. $ac - 2c + 3a - 6$.
17. $10 - 5y + 2x - xy$.
19. $x^3 - x + 2x^2 - 2$.

12. $a^3 - a^2b - ab^2 + b^3$.
14. $ax^2 - x + ax - 1$.
16. $m^3 + pn + m^2n + mp$.
18. $ax + bx + cx + a + b + c$.
20. $c^2d - cd^2 + bcd + ac - ad + ab$.

226. Sum and Difference of Two Cubes.

By multiplication,

1. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$.
2. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

From (1), it is evident that one of the factors of $a^3 + b^3$ is $a + b$, and by the division of $a^3 + b^3$ by $a + b$, we obtain the other factor, $a^2 - ab + b^2$.

Hence, the factors of the sum of the cubes of two numbers are a binomial factor, which is the sum of the two numbers, and a trinomial factor which is the sum of the squares of the numbers minus their product.

From (2) it is evident that one of the factors of $a^3 - b^3$ is $a - b$, and by the division of $a^3 - b^3$ by $a - b$, we obtain the other factor, $a^2 + ab + b^2$.

Hence, the factors of the difference of the cubes of two numbers are a binomial factor, which is the difference of the two numbers, and a trinomial factor, which is the sum of the squares of the numbers plus their product.

Illustrations.

1. $8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9)$.
2. $125c^3 + 1 = (5c + 1)(25c^2 - 5c + 1)$.

PROBLEMS

Find the products:

1. $(y - x)(y^2 + yx + x^2)$. $y^3 - x^3$
3. $(y - 1)(y^2 + y + 1)$.

Factor:

2. $y^3 - x^3$.
4. $y^3 - 1$.

Find the products of the following:

5. $(a - c)(a^2 + ac + c^2)$.
6. $(y - 3)(y^2 + 3y + 9)$.
7. $y(y + 1)(y^2 - y + 1)$.
8. $(x^2 - b^2)(x^4 + x^2b^2 + b^4)$.
9. $m^2(1 - m)(1 + m + m^2)$.
10. $(2a - 3b)(4a^2 + 6ab + 9b^2)$.
11. $ab(a + b)(a^2 - ab + b^2)$.
12. $(x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$.
13. $(3a - 4b)(9a^2 + 12ab + 16b^2)$.
14. $\left(x - \frac{1}{4}\right)\left(x^2 + \frac{x}{4} + \frac{1}{16}\right)$.
15. $(m^2 + 5)(m^4 - 5m^2 + 25)$.
16. $(y - x^3)(y^2 + yx^3 + x^6)$.

Factor the following:

17. $a^3 + c^3$.
18. $x^3 + 27$.
19. $x^4 + x$.
20. $8a^3 - 27b^3$.
21. $a^4b - ab^4$.
22. $8a^3 + 27b^3$.
23. $x^6 + y^6$.
24. $a^2 - a^5$.
25. $x^6 - y^6$.
26. $27a^3 + 64b^3$.
27. $x^3 + \frac{1}{64}$.
28. $m^6 - 125$.

227. Summary.—Practically all the expressions can be factored by using one or more of the foregoing types of factoring.

First, factor out a monomial factor wherever possible. Then in factoring a binomial expression consider if it is of anyone of these three types.

1. $x^2 - y^2 = (x - y)(x + y)$.
2. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
3. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

In factoring a trinomial expression consider if it comes under one of the following types.

1. $x^2 + 2xy + y^2 = (x + y)(x + y)$.
2. $x^2 - 2xy + y^2 = (x - y)(x - y)$.
3. $x^2 + (b + c)x + bc = (x + b)(x + c)$.
4. $ax^2 + bx + c = (mx + p)(nx + r)$.
5. $x^4 + bx^2y^2 + y^4 = (x^2 + dxy + y^2)(x^2 - dxy + y^2)$.

If the expression to be factored is a polynomial of more than three terms use the method by grouping, re-arranging terms if necessary, or use §232.

228. Test of Divisibility.

By actual division,

I. $(x^2 + y^2) \div (x + y) = x - y$, Rem., $2y^2$.
$(x^3 + y^3) \div (x + y) = x^2 - xy + y^2$.
$(x^4 + y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3$, Rem., $2y^4$.
$(x^5 + y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

Note that in the above illustrations only the **sum** *of the* **same odd** *powers of two numbers is divisible by the* **sum** *of the numbers.*

II. $(x^2 + y^2) \div (x - y) = x + y$, Rem., $2y^2$.
$(x^3 + y^3) \div (x - y) = x^2 + xy + y^2$, Rem., $2y^3$.
$(x^4 + y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3$, Rem., $2y^4$.
$(x^5 + y^5) \div (x - y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4$, Rem., $2y^5$.

Note that in the above illustrations the **sum** *of the* **same** *powers of two numbers is* **never** *divisible by the* **difference** *of the numbers.*

III. $(x^2 - y^2) \div (x + y) = x - y$.
$(x^3 - y^3) \div (x + y) = x^2 - xy + y^2$, Rem., $-2y^3$.
$(x^4 - y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3$.
$(x^5 - y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$, Rem., $-2y^5$.

Note that in the above illustrations only the **difference** *of the same* **even** *powers of two numbers is divisible by the* **sum** *of the numbers.*

IV. $(x^2 - y^2) \div (x - y) = x + y$.
$(x^3 - y^3) \div (x - y) = x^2 + xy + y^2$.
$(x^4 - y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3$.
$(x^5 - y^5) \div (x - y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

Note that in the above illustrations the **differences** *of the same powers of two numbers is* **always** *divisible by the* **difference** *of the numbers.*

229. Laws of Signs.

1. When $x + y$ is the divisor the signs of the terms in the quotient are alternately plus and minus.

2. When $x - y$ is the divisor the signs of all the terms in the quotient are plus.

230. Laws of Exponents.—In the quotient the exponent of **x** decreases and that of **y** increases by 1 in each succeeding term.

231. Generalization.—When **n** is any positive integer, §228 may be stated more briefly and in general terms as follows:

I. $a^n + b^n$ is exactly divisible by $a + b$ if **n** is **odd.**
II. $a^n + b^n$ is never exactly divisible by $a - b$.
III. $a^n - b^n$ is exactly divisible by $a + b$ if **n** is **even.**
IV. $a^n - b^n$ is always divisible by $a - b$.

REVIEW PROBLEMS

Find the prime factors of the following, removing a monomial factor first wherever possible.

1. $5x^2 - 25x + 30$.
2. $ay^6 + 6ay^3 + 9a$.
3. $x^3 - 64m^3$.
4. $4x^3 + 32$.
5. $m^2 + 12m - 45$.
6. $4x^2 - 32x + 64$.
7. $x^2y^2 + 17xy - 60$.
8. $m^3 - mn^2$.
9. $24x^2 + 46x - 18$.
10. $3x^6 - 3y^6$.
11. $b^2 + 24b + 143$.
12. $a^6 + 2a^3 - 99$.
13. $64x^6 + 27y^6$.
14. $x^2 - \frac{2x}{3} + \frac{1}{9}$.
15. $5x^8 - 5y^8$.
16. $16y^4 + 40y^2 + 25$.
17. $x^{12} - y^{12}$.
18. $x^{14} - 2x^7y + y^2$.
19. $169x^2 + 104xy + 16y^2$.
20. $ac - cx - ad + dx$.
21. $10a^2 - a - 3$.
22. $6b^2 + 13bc - 63c^2$.
23. $6c^2 - 43c + 72$.
24. $6c^2 - 11c - 72$.
25. $6c^2 + 11c - 72$.
26. $6c^2 + 43c + 72$.
27. $25a^2 - 90ab + 81b^2$.
28. $24ax^3 - 81ay^3$.
29. $y - 16 + 15y^2$.
30. $x^2 + 11x - 126$.
31. $36 + x^2 - 12x$.
32. $169x^2 - \frac{a^2}{4}$.
33. $\frac{x^2}{9} - 10x + 225$.
34. $4x^4 - \frac{1}{81}$.
35. $16x^4 - \frac{1}{81}$.
36. $5a^4 - 5b^4$.
37. $a^2 + 17a - 270$.
38. $6x^2 - 11x - 182$.
39. $16x^2 - b^2$.
40. $m^2 + 5m - 14$.
41. $m^6 - 1$.
42. $-28 + m^2 - 3m$.
43. $b^2 + 12b + 32$.
44. $b^2 - 4b - 32$.
45. $k^2 - 2k - 99$.
46. $ac - bd^2 + bc - ad^2$.
47. $p^2 - 6p - 7$.
48. $x^5 + y^5$.
49. $x^5 - y^5$.
50. $m^4 - 16n^6$.
51. $a^8 - b^8$.
52. $6x^2 - 7x - 3$.
53. $16x^2 - 4y^4$.
54. $12m^2 + 7m - 12$.
55. $a^2 - 2ab + b^2 - c^2$.
56. $m^2 - n^2 + 2ma + a^2$.
57. $4b^2 + 8b + 3$.
58. $4x^2 - 40xm + 100m^2$.
59. $24x^2 - 28x - 12$.
60. $60m^2 + 35m - 60$.
61. $3x^5 - 3y^5$.
62. $7p^2 - 42p - 49$.
63. $24y^3x + 81x^4$.
64. $169x^2m - 4y^2m$.
65. $c^2 + 9c - 52$.
66. $169b^2 + 78xb + 9x^2$.
67. $25y^2 + 60xy + 36x^2$.
68. $x^2 + \frac{1}{3}x + \frac{1}{36}$.
69. $3d^2 - 15d - 42$.
70. $-12x + x^2 + 36$.
71. $1 + 32m^5$.
72. $x^{10} - y^{10}$.
73. $x^6 + y^6$.
74. $a^2 + 2ab + b^2 - x^2$.
75. $x^2 - (m - n)^2$.
76. $x^2 - y^2 + 2yz - z^2$.
77. $(a + x)^2 + 23(a + x) + 60$.
78. $(m - n)^2 + 2(m - n) - 15$.
79. $(x + y)^3 - m^3$.
80. $(r + s)^3 + 27t^3$.
81. $a^2 - 2ab + b^2 - c^2 - 2cd - d^2$.
82. $(x^2 + x - 9)^2 - 9$.
83. $m^7 + n^7$.

232. The Remainder or Factor Theorem.—The remainder or factor theorem gives a method of finding a divisor, or factor, of an expression without actually going through the process of division.

Illustration 1. By actual division,

$$\begin{array}{llll|l}
x^2 & -5x & & +6 & x - a \\
\cline{5-5}
x^2 - ax & & & & x + a - 5 \\
\cline{1-1}
 & ax - 5x & & & \\
 & ax & -a^2 & & \\
\cline{1-4}
 & -5x + a^2 & & +6 & \\
 & -5x & +5a & & \\
\cline{2-4}
 & & a^2 - 5a + 6 \text{ Rem.} & &
\end{array}$$

Note.—*The remainder is the same as the dividend except that every* **x** *is replaced by* **a**. *This means that if $x - a$ is an exact divisor the remainder is zero.*

Illustration 2. By actual division,

$$\begin{array}{l|l} x^3 - 2x^2 - 3x + 8 & x - 2 \\ \underline{x^3 - 2x^2} & \overline{x^2 - 3} \\ \qquad\quad -3x + 8 & \\ \qquad\quad \underline{-3x + 6} & \\ \qquad\qquad\quad 2 \text{ Rem.} & \end{array}$$

Substituting 2 for x in

$x^3 - 2x^2 - 3x + 8$, we have:

$2^3 - 2(2)^2 - 3(2) + 8 =$

$8 - 8 - 6 + 8 = 2$

Illustration 3. By actual division,

$$\begin{array}{l|l} x^3 + 2x^2 - 3x - 8 & x + 2 \\ \underline{x^3 + 2x^2} & \overline{x^2 - 3} \\ \qquad\quad -3x - 8 & \\ \qquad\quad \underline{-3x - 6} & \\ \qquad\qquad -2 \text{ Rem.} & \end{array}$$

Substituting (-2) in

$x^3 + 2x^2 - 3x - 8$, we have:

$(-2)^3 + 2(-2)^2 - 3(-2) - 8 =$

$\cancel{-8 + 8} + \cancel{6} - \cancel{8} = -2.$

Note. *Here (-2) is substituted in the expression to be tested and not $(+2)$. In illustration 2, $(+2)$ is substituted and not (-2). In illustration 1, $(+a)$ is substituted and not $(-a)$.*

PROBLEMS

Find the remainder in each of the following by the above method.

1. $\dfrac{x^3 + 2x^2 - x - 6}{x + 2}$.
2. $\dfrac{x^3 - x^2 - 7x + 5}{x - 3}$
3. $\dfrac{x^3 + 6x^2 + 5x - 3}{x + 3}$.
4. $\dfrac{x^3 + 2x^2 - x - 2}{x + 2}$.

233. Abbreviated Division.—Abbreviated or synthetic division is a short method of division in which we find both quotient and remainder. The divisor must be of the form $x - a$. The remainder may be zero.

(1)
$$\begin{array}{l|l} x^3 + 4x^2 + x - 8 & x + 2 \\ \underline{x^3 + 2x^2} & \overline{x^2 + 2x - 3} \\ \qquad 2x^2 + x & \\ \qquad \underline{2x^2 + 4x} & \\ \qquad\quad -3x - 8 & \\ \qquad\quad \underline{-3x - 6} & \\ \qquad\qquad\quad -2 & \end{array}$$

$\therefore$ Quotient $= x^2 + 2x - 3$ and Rem. $= -2$.

(2)
$$\begin{array}{l|l} \text{Dividend} & \text{Divisor} \\ 1 + 4 + 1 - 8 & \underline{-2} \\ \underline{\quad -2 - 4 + 6} & \end{array}$$

Quotient $= 1 + 2 - 3 \mid -2 =$ Remainder.

The bar ($|$) separates the quotient from the remainder.

We are familiar with method (1). Now study method (2) where we have performed the same division but have omitted the letter bases and used only the numerical coefficients.

Explanation.

1. We change the divisor to -2 instead of $x + 2$. Be careful to change the sign of the 2nd term of the divisor.

2. The coefficient of the first term in the quotient is the same as that in the dividend.

3. Multiply the divisor by 1 of the quotient and put -2 under 4 of the dividend. **Now add.** (**Do not subtract** as we have changed the sign of $+2$ of the divisor to -2 and this allows us to add rather than subtract.)

4. Now multiply the divisor by $+2$ of the quotient and put -4 under 1 of the dividend and -3 in the quotient.

5. Multiply the divisor by -3 of the quotient and get $+6$ under -8 of the dividend and add getting -2 for the remainder.

Practice this method of division in the following problems and you will learn to save time in divisions of this type.

Note.—*To apply abbreviated division to an expression as* $b^3 + 1$, *it must be written* $b^3 + 0 \cdot b^2 + 0 \cdot b + 1$. *When the coefficient of a term is zero, that term does not appear in the expression.*

PROBLEMS

Factor (Use Arts. 232 and 233):

1. $x^3 + 3x^2 - 9x + 5$.
2. $a^3 + 3a^2 + 8a + 12$.
3. $m^3 + m - 2$.
4. $y^3 - y^2 + 5y - 5$.
5. $m^3 + 2m^2 + mn + 2n$.
6. $2x^3 - x - 1$.
7. $2x^3 - x^2 - 13x - 6$.
8. $x^3 + 6x^2 + 11x + 6$.
9. $m^3 + 2m^2 - 16$.
10. $2x^3 - 4x^2 - 7x + 6$.
11. $x^3 - 2x^2 - 23x + 60$.
12. $x^3 + x^2 - x - 1$.
13. $x^4 - 13x^3 + 21x^2 - 13x - 22$.
14. $x^4 + 7x^3 + 8x^2 - 28x - 48$.

234. Application of Factoring to the Solution of Equations.

A common method of solving an equation of the type $ax^2 + bx + c = 0$ is by factoring.

Illustrations 1. Solve $x^2 - x - 6 = 0$.

Solution.—Factoring the first member of the equation we get $(x + 2)(x - 3) = 0$.

If the product of two or more factors equals zero, then one of the factors must equal zero. It must be understood that any one of the factors may equal zero.

Hence, to obtain the roots of the above equation, let each factor in turn equal zero, and solve the resulting simple equations. The number of roots is the same as the highest power of the unknown number.

If $x + 2 = 0$,
Then $x = -2$,
Check for $x = -2$.
$4 + 2 - 6 = 0$.

and if $x - 3 = 0$.
then $x = 3$.
Check for $x = 3$.
$9 - 3 - 6 = 0$.

2. Solve $3x^2 - 4x = 4$.

Transposing, $3x^2 - 4x - 4 = 0$.

Factoring, $(3x + 2)(x - 2) = 0$.

Hence, if $3x + 2 = 0$ and if $x - 2 = 0$.

Then $x = -\frac{2}{3}$ then $x = 2$.

Check for $x = -\frac{2}{3}$. Check for $x = 2$.

$3(\frac{4}{9}) - 4(-\frac{2}{3}) = 4$. $3(4) - 8 = 4$.

$\frac{4}{3} + \frac{8}{3} = 4$. $12 - 8 = 4$.

PROBLEMS

Solve and check the following:

1. $x^2 - 9x + 20 = 0$.
2. $x^2 - x - 2 = 0$.
3. $x^2 + x = 6$.
4. $x^2 - 25 = 0$.
5. $x(x + 7) + 12 = 0$.
6. $x^2 - 11x + 28 = 0$.
7. $x^3 + x^2 - 16x = 16$
8. $2x^3 + x^2 - 5x + 2 = 0$.
9. $3x^2 + 4x = 4$.
10. $25y^2 - 49 = 0$.
11. $3y^3 + 5y^2 - 11y + 3 = 0$.
12. $m^4 - 4m^2 = 0$.
13. $x^3 - x^2 - 4x + 4 = 0$.
14. $3(x^2 - 1) - 2(x + 1) = 0$.
15. $2x^4 + 5x^3 - x^2 - 6x = 0$.

16. The square of a certain number diminished by the number equals 2. Find the number.

17. The square of a certain number increased by 3 times the number equals 28. Find the number.

$x^3 = 36x$

18. The cube of what number equals 36 times the number?

19. What number plus its square equals 6?

20. Find two consecutive numbers whose product is 12.

21. The length of a lot is 5 yards more than the width. Its area is 104 square yards. Find the length and width.

22. If to 3 times the square of a certain number twice the number is added the result equals 85. Find the number.

23. What is the width of a walk surrounding a rectangular plot 50 feet by 30 feet, if the walk contains 704 square feet?

235. **Highest Common Factor** (H. C. F.).—The highest common factor of two or more expressions is the highest divisor common to each of the expressions. It is the product of all the common factors, each common factor being used the least number of times it occurs in any expression.

Illustrations. 1. Find the H. C. F. of 12, 4, 8. It is evident that 2 and 4 will divide each of them exactly. Therefore 2 and 4 are both factors of 12, 4, 8, but 4 is greater than 2 and is the largest number contained in each of these three numbers. Therefore, 4 is the H. C. F. of 12, 4, 8.

2. Find the H. C. F. of $3a^2b$, $6ab^2$, and $15a^2b^2$. It is seen that $3ab$ is contained in $3a^2b$, **a** times; in $6ab^2$, **2b** times, and in $15a^2b^2$, **5ab** times. Hence, $3ab$ is the highest number, for we can find no other divisor contained in the quotients **a, 2b** and **5ab.**

3. Find the H. C. F. of $5x^2y^3$, $15x^2y^3$, and $20x^3y^5$.

1. 5 is the highest divisor of all the numerical coefficients.
2. x^2 is the highest power of x contained in each.
3. y^3 is the highest power of y contained in each.

Then $5x^2y^3$ is the H. C. F. of $5x^2y^3$, $15x^2y^3$, and $20x^3y^5$.

Therefore, to find by inspection the H. C. F. of two or more monomial expressions proceed as follows:

1. *Find the greatest number which will divide exactly all the numerical coefficients.*
2. *Choose the highest power of each literal factor used in each expression.*
3. *Write all these factors as a product.*

236. **Highest Common Factor of Polynomials.**

Problem.—Find the H. C. F. of $8x^2 - 28x + 24$ and $12x^2 + 16x - 36$.

Solution.—Factor each expression and arrange in a column as shown.

$$\begin{aligned} 8x^2 - 28x + 24 &= 4(x - 2)(2x - 3). \\ 12x^2 + 6x - 36 &= 6(x + 2)(2x - 3). \end{aligned}$$

Now consider $4(x - 2)(2x - 3)$ and $6(x + 2)(2x - 3)$ as monomial expressions and by inspection find their H. C. F. §234.

Therefore, since 2 is the greatest divisor of the numerical coefficients and $(2x - 3)$ is the only other factor common to both, we have the H. C. F. of $8x^2 - 28x + 24$ and $12x^2 + 6x - 36$ is $2(2x - 3)$.

PROBLEMS

Find the H. C. F. of the following:

1. $(x - 2)(x - 3)$ and $(x - 2)(x + 4)$.
2. $x^2 + 2x$ and $x^3 + x$.
3. $2x^2 + 4x$ and $3x^2 + 6x$.
4. $x^2 - 5x + 6$ and $x^2 - 4$.
5. $x^2 - x - 6$ and $x^2 - 4$. ✓
6. $a^2b^3 + ab^2$ and $a^3b^4 + a^2b^2$. A+B a+d
7. $x^2 - y^2$ and $x^2 + 2xy + y^2$.
8. $ac + ad + bc + bd$ and $ac - ad + bc - bd$.
9. $a^2 - 2ab + b^2$ and $a^3 - b^3$.
10. $x^2 - 9$, $x^2 - 6x + 9$, $x^2 + x - 12$.
11. $a^2 - 4a + 4$, $a^2 - 4$, $a^2 + a - 6$.
12. $x^2 - 3x + 2$, $x^2 + x - 6$, $x^2 - 6x + 8$.

237. **Multiple.**—A multiple of a number is any number which will contain the given number without a remainder. Thus, 8, 12, 16, 20 are all multiples of 4, since 4 will divide into each of them without a remainder.

238. **Common Multiple.**—A common multiple of two or more numbers is a multiple of each of them. Thus 12 is a multiple of 2, 3, 4, 6, and 12, since each of these numbers can be divided into 12 exactly.

239. **Least Common Multiple** (L. C. M.).—The least common multiple of two or more expressions is the product of all the different factors, each factor being used the greatest number of times it occurs in any expression.

Illustrations.—1. 3, 4, and 5 can each be divided into 60. Therefore 60 is a common multiple of 3, 4, and 5. Again, 3, 4, and 5 can each be divided into 120, but 60 is less than 120 and therefore 120 is not the least common multiple, while 60 is.

2. By inspection, the L. C. M. of $3a^2b^3$ and $5a^4b$ is $15a^4b^3$.

240. **The Least Common Multiple of Polynomials.**

Problem. Find the L. C. M. of $2x - 16$, $x^2 - 2x - 48$, and $x^2 + 12x + 36$.

Solution.—Factor each expression and arrange in a column as shown.

1. $2x - 16 = 2(x - 8)$.
2. $x^2 - 2x - 48 = (x - 8)(x + 6)$.
3. $x^2 + 12x + 36 = (x + 6)(x + 6)$.

The L. C. M. of any number of expressions must contain all the factors of each expression. Therefore, to contain (1) we need $2(x - 8)$; to contain (2) we need $(x - 8)$ and $(x + 6)$. But we have the factor $(x - 8)$ and need to take from (2) only the factor $(x + 6)$, giving $2(x - 8)(x + 6)$. To contain (3) we need $(x + 6)$ twice as a factor. In $2(x - 8)(x + 6)$ we have $(x + 6)$ once, therefore we need one factor $(x + 6)$ from (3). Finally the L. C. M. of (1), (2), and (3) is $2(x - 8)(x + 6)(x + 6)$ or $2(x - 8)(x + 6)^2$.

To find the L. C. M. of any number of expressions:

1. *Arrange all the expressions in a column.*
2. *Find the prime factors of each expression.*
3. *Select* all *the factors of the first expression, and* all *the factors of each expression following which have not already been used as factors, from the preceding expressions.*
4. *Write these factors as a product. This product is the L. C. M. of all the given expressions.*

PROBLEMS

Find the L. C. M. of the following:

1. a^2b, ab^3, and a^3b^2.
2. $x + 2$ and $x^2 - 4$.
3. $a^2 + ab$ and $a^3 + a^2b$.
4. $x^2 - 2xy + y^2$ and $x^2 - y^2$.
5. $3x - 6$, $2x - 4$, and $x^2 - 3x + 2$.
6. $x^2 - x - 6$ and $x^2 + x - 2$.
7. $x^2 - 7x + 12$ and $x^2 - 2x - 3$.
8. $x^2 + 2xy + y^2$ and $x^2 - y^2$.
9. $m^2 + mn - 2n^2$ and $m^2 - mn$.
10. $a^3 + b^3$ and $a^2 - ab + b^2$.
11. $x^2 - x^2y$, $xy - xy^2$, and $x^2 + xy^2 - x^2y - xy$.
12. $m^2 - m - 6$ and $m^2 - m - 2$.
13. $x^3 - x^2 + x$, $x^2 + x$, and $x^3 + 1$.
14. $6x^2 - 17xy + 12y^2$ and $12x^2 - xy - 20y^2$.

CHAPTER VI

FRACTIONS

241. **Importance of Fractions**—A great many engineering problems involve the solution of equations containing simple and complex fractions, as well as mixed expressions. The engineer must be able to simplify and solve these expressions either mentally or by rapid calculations.

242. **Integer.**—An integer is a whole number as, 5, 9, 27, ax, $bx + cy$, etc.

243. **Fraction.**—In Arithmetic we learned that a fraction is one or more of the equal parts of a whole number, or unit, as $\frac{1}{3}$, $\frac{3}{4}$, $\frac{6}{5}$, etc. It is now necessary to extend the idea of a fraction and consider it as an indicated quotient. Thus $\frac{1}{3}, \frac{6}{5}, \frac{a}{b}, \frac{xy}{z}$, etc. In the fraction $\frac{a}{b}$, **a** is the **numerator** and **b**, the **denominator**. The numerator and denominator are called the **terms** of the fraction. $\frac{a}{b}$ is read **a** over **b**, or **a** divided by **b**.

244. **Proper Fraction.**—A proper fraction is a fraction whose numerator is less than or of lower degree than the denominator.

Thus, $\frac{2}{3}, \frac{4}{7}, \frac{a^2}{a^3}, \frac{xy^2}{x^2y^3}$, are proper fractions. In $\frac{a^2}{a^3}$, the exponents 2 and 3 indicate the degree of **a**.

245. **Improper Fraction.**—An improper fraction is a fraction whose numerator is of a degree equal to or greater than that of the denominator.

Thus, $\frac{4}{3}, \frac{9}{7}, \frac{a^2b^2}{ab}, \frac{m^3n^4}{m^2n}$, are improper fractions.

Note.—*The fraction,* $\frac{m^2n^3}{m + n^4}$, *is said to be improper with respect to* **m** *and proper with respect to* **n**.

246. **Principles of Fractions.**—Given the fractional form $\frac{6}{2}$ which equals 3. If we multiply both numerator and denominator by 2 we get $\frac{12}{4}$ which also equals 3. If we divide both numerator and denominator by 2 we get $\frac{3}{1}$ or 3.

It is evident from the above that multiplying or dividing both numerator and denominator of a fraction by the same number does **not change** the value of the fraction.

It follows that $\frac{a + b}{c - d} \cdot \frac{n}{n} = \frac{an + bn}{cn - dn}$, Why?

and also that $\frac{mn - n^2}{pn + qn} = \frac{n(m - n)}{n(p + q)} = \frac{m - n}{p + q}$. Why?

247. I. **Signs of a Fraction.**—The sign before the fraction is the sign of the fraction, as $-\frac{2}{3}$. In addition to this sign, the numerator and denominator each have a sign, as $-\frac{+a}{-b}$. The signs of the numerator and denominator combine according to the law of signs in division.

Illustrations.—By arithmetic numbers we have,

1. $+\frac{+15}{+5} = +3$, for $+[(+15) \div (+5)] = +(+3) = +3$.
2. $+\frac{+15}{-5} = -3$, for $+[(+15) \div (-5)] = +(-3) = -3$.
3. $+\frac{-15}{+5} = -3$, for $+[(-15) \div (+5)] = +(-3) = -3$.
4. $+\frac{-15}{-5} = +3$, for $+[(-15) \div (-5)] = +(+3) = +3$.

II. **Laws of Signs of Fractions.**—By comparing each illustration with the first we derive the following laws of signs.

(*a*) *If the sign of the numerator or the denominator of a fraction is changed the sign of the* **value** *of the fraction* **is changed.**

(*b*) *If the signs of both the numerator and the denominator of a fraction are changed the sign of the* **value** *of the fraction is* **not changed.**

By algebraic numbers we have,

1. $+\frac{+x}{+y} = +\frac{x}{y}$.   2. $+\frac{+x}{-y} = -\frac{x}{y}$.

3. $+\frac{-x}{+y} = -\frac{x}{y}$.   4. $+\frac{-x}{-y} = +\frac{x}{y}$.

Note.—*When the numerator or the denominator of a fraction is made up of two or more* **factors,** *the sign of that term is* **changed** *if the signs of an* **odd** *number of* **factors** *are changed. The sign of that term is* **not changed** *if the signs of an* **even** *number of* **factors** *are changed. This follows from the law of signs in multiplication.*

Thus, $$\frac{(a-b)(c-d)}{(e-f)(g-h)} = -\frac{(a-b)(d-c)}{(e-f)(g-h)} \text{ or } \frac{(a-b)(d-c)}{(f-e)(g-h)},$$

and $$\frac{(x-y)(x-y)}{(p-q)(r-s)} = \frac{(y-x)(y-x)}{(p-q)(r-s)} \text{ or } -\frac{(x-y)(y-x)}{(q-p)(s-r)}$$

Prove each of the above equalities by multiplying out each numerator and denominator.

PROBLEMS

Study carefully the following and determine which are true. Give reasons.

1. $-\frac{+a}{+b} = +\frac{-a}{+b}$.   2. $\frac{a}{b} = \frac{-a}{-b}$.

3. $-\frac{a}{b} = \frac{a}{-b}$.   4. $-\frac{m}{m-n} = \frac{m}{n-m}$.

5. $-\frac{m}{m-n} = \frac{m}{m-n}$.   6. $\frac{b}{(a-c)(x-y)} = \frac{-b}{(c-a)(x-y)}$.

248. **Lowest Terms of a Fraction.**—A fraction is in its lowest terms when the numerator and denominator have no common factor.

Illustrations.—1. $\frac{12}{18} = \frac{\cancel{2} \times 2 \times \cancel{3}}{\cancel{2} \times 3 \times \cancel{3}} = \frac{2}{3}$. The fraction, $\frac{12}{18}$ is not expressed in lowest terms for we can divide out factors common to both the *numerator and the denominator* and get the resulting equal fraction $\frac{2}{3}$.

2. $\frac{a^2b^3}{ab^4} = \frac{a}{b}$, thus, $\frac{a^2b^3}{ab^4}$ is not expressed in lowest terms, but reduces to $\frac{a}{b}$.

Hence to reduce a fraction to its lowest terms, factor both numerator and denominator and divide out the common factors.

3. $\frac{a^2 + 5a + 6}{a^2 - a - 6} = \frac{(a + 3)(a + 2)}{(a - 3)(a + 2)} = \frac{a + 3}{a - 3}$.

PROBLEMS

Reduce the following fractions to lowest terms: Check as in §31

1. $\frac{ab}{bc}$.
2. $\frac{3m^2n}{18mn^2}$.
3. $\frac{-a^9}{a^{11}}$.
4. $\frac{6kr^2s}{-3kr^2s^2}$.
5. $\frac{-12x^2y^3z^4}{14x^3y^2z^5}$.
6. $\frac{17a^3bc^3d}{51ab^2cd}$.
7. $\frac{4\pi r^2}{8\pi rn}$.
8. $\frac{x(x - y)}{y(x - y)}$.
9. $\frac{(x - y)(x + y)}{(x - y)(x - y)}$.
10. $\frac{a^2 - b^2}{a^3 - b^3}$.
11. $\frac{a^2 - ab + b^2}{a^3 + b^3}$.
12. $\frac{a^2 - 2ab + b^2}{a^3 - b^3}$.
13. $\frac{a(a^2 - 1)}{(a + 1)(a + 1)}$.
14. $\frac{x^5 - x^2}{x^2 - 2x + 1}$.
15. $\frac{(a - 1)(a + 2)}{(a + 2)(a - 3)}$.
16. $\frac{a(a^2 - 4)(a - 3)}{a^3(a^2 - a - 6)}$.
17. $\frac{ac + bd + ad + bc}{ax - ay + bx - by}$.
18. $\frac{2x^2 + 7x - 15}{x^2 - 2x - 35}$.
19. $\frac{3x^2 + 20x - 63}{4x^2 + 25x - 99}$.
20. $\frac{20 - 12x}{10 + 19x - 15x^2}$.
21. $-\frac{a^2bx^2}{ax^2 - axy}$.
22. $-\frac{x^4 - y^4}{x^6 - y^6}$.
23. $-\frac{ax^3 - ay^3}{mx^2 + mxy + my^2}$.
24. $\frac{x^3 + x^2 - x - 1}{x^3 - x^2 + x - 1}$.

MULTIPLICATION OF FRACTIONS

249. **Process.**—The process of multiplication of fractions in algebra is the same as in arithmetic.

Illustrations.

1. In arithmetic, $\frac{2}{3} \times 2 = \frac{4}{3}$. 2. In algebra, $\frac{a}{b} \times c = \frac{ac}{b}$.

3. In arithmetic, $\frac{\cancel{2}}{3} \times \frac{5}{\underset{2}{\cancel{4}}} = \frac{5}{6}$. 4. In algebra, $\frac{a}{\cancel{b}} \times \frac{\overset{b}{\cancel{b^2}}}{c} = \frac{ab}{c}$.

5. Again, $\frac{4ac^2d}{5bd^3} \times \frac{3abd^2}{2bcd^2} = \frac{6a^2c}{5bd^2}$.

6. Also, $\frac{(x + y)}{(x - y)} \times \frac{x^2 - 2xy + y^2}{x^2 - y^2}$

$$= \frac{\cancel{(x + y)}}{\cancel{(x - y)}} \times \frac{\cancel{(x - y)}\cancel{(x - y)}}{\cancel{(x + y)}\cancel{(x - y)}} = \frac{1}{1} = 1$$

Therefore.

1. *To multiply a fraction by an integer, multiply the numerator only by that integer.*

2. *To multiply a fraction by a fraction, divide out all the like factors in the numerators and denominators. Multiply together all the remaining factors in the numerators for the numerator of the product and all the remaining factors of the denominators for the denominator of the product.*

250. Powers of a Fraction.

By multiplication,

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2} \text{ and } \left(\frac{a+b}{c-d}\right)^2 = \frac{a+b}{c-d} \times \frac{a+b}{c-d} = \frac{(a+b)^2}{(c-d)^2}.$$

Therefore, any given power of a fraction is the given power of the numerator over the same power of the denominator.

PROBLEMS

Find the values of the following: Check as in §31.

1. $\frac{2}{x} \cdot \frac{b}{y}$.
2. $\frac{1}{a} \times \frac{3}{b}$.
3. $\frac{3x}{4y} \times \frac{5y^2}{6x^3}$.
4. $m \cdot \frac{m^2}{mn}$.
5. $\frac{7x^2y}{3xy^2} \times \frac{4x^2y^3}{12xy^5}$.
6. $\frac{-7bx}{12ax} \times \frac{6bxy^2}{3by^4}$.
7. $\frac{3a^2}{4b} \times \frac{5b^2}{3a} \times \frac{12ab}{10a^2b}$.
8. $\frac{21x^2}{5xy} \times \frac{35x^3}{7y^2} \times \frac{14xy^2}{3x^4}$.
9. $\frac{x^2 - ax}{3x + 3a} \times \frac{3}{2}$.
10. $\frac{4(x-2)}{x^2 - 4} \times \frac{6(x^2 + 4x + 4)}{12(x + 2)}$.
11. $\frac{3(a - b)}{a^3 - b^3} \times \frac{a^2 + ab + b^2}{6(a^2 - b^2)}$.
12. $\frac{a^2 - 5a + 6}{a^2 - 9} \times \frac{a^2 - 3a}{a^3 - 2a^2}$.
13. $\frac{x^3 + xy^2}{x^4 - y^4} \times \frac{x^2 - y^2}{x + y}$.
14. $\frac{5m - 10}{6m^2 - 24} \times \frac{18m + 36}{35m - 70}$.
15. $\frac{12p^2 - 14pq + 2q^2}{4p^2 + 7pq - 2q^2} \times \frac{8p^2 + 2pq - q^2}{12p^2 - 27pq + 6q^2}$.
16. $\frac{ac + bc + ad + bd}{ac + bd - bc - ad} \times \frac{3ac - 3bc + 3bd - 3ad}{4ac - 4bd + 4ad - 4bc}$.

DIVISION OF FRACTIONS

251. Division of a Fraction by an Integer.—To divide a fraction by an integer, divide the numerator of the fraction by that integer. Thus, $\frac{4}{5} \div 2 = \frac{2}{5}$. This is the same as multiplying the fraction by the reciprocal of the integer.

Thus, $\frac{4}{5} \div 2 = \frac{4}{5} \times \frac{1}{2} = \frac{2}{5}$.

Note.—The reciprocal of a number is 1 divided by that number. Thus the reciprocal of 2 is $\frac{1}{2}$.

252. Division of a Fraction by a Fraction.—To divide a fraction by a fraction invert the divisor and proceed as in multiplication. Since an integer may be considered a fraction with 1 as a denominator, we have the general method for division of fractions as follows:

1. *Factor all terms of the given fraction which are not prime.*
2. *Invert the divisor.*
3. *Proceed as in multiplication.*

PROBLEMS

Perform the indicated operations: Check as in §31.

1. $\frac{6}{5} \div \frac{8}{5}$.
2. $\frac{12}{18} \div \frac{3}{36}$.
3. $\frac{11}{18} \div \frac{22}{9}$.
4. $\frac{7c^2}{3d} \div \frac{4c^3}{d^2}$.
5. $\frac{21m^2n}{5mn} \div \frac{7mn^2}{n}$.
6. $\frac{18abc}{3ac} \div \frac{12b^2ac}{3ca}$.
7. $\frac{mn^2p}{pm} \div mn$.
8. $pq^2 \div \frac{p^3}{q}$.
9. $\frac{3x}{x^2 - 2x} \div \frac{x}{x - 2}$.
10. $\frac{2r - 3}{r^2 - 4} \div \frac{r - 2}{r + 2}$.
11. $\frac{2m - 5}{4m^2 - 9} \div \frac{2m - 5}{2m - 3}$.
12. $\frac{c^4 - d^4}{c + d} \div \frac{c^2 - d^2}{c - d}$.
13. $\frac{x^2 - y^2}{(x - y)^2} \div \frac{x + y}{x - y}$.
14. $\frac{2r^2 - 4r}{3r^2 + 9r} \div \frac{4r}{3r}$.
15. $\frac{a^3 - 2a^2 + a}{a^2 + 2a + 1} \div \frac{(a - 1)^2}{a + 1}$.
16. $\frac{a^3 - 8}{a + 2} \div \frac{a - 2}{a^2 + 4a + 4}$.
17. $\frac{a^2 - 5a - 14}{a^2 + 2a - 8} \div \frac{a^2 + a - 56}{a^2 + 5a + 4}$.
18. $\frac{x^3 + 27}{x + 2} \div \frac{x^2 - 6x - 27}{x^2 - 4}$
19. $\frac{m^3 + m^2 - 6m}{m^2 + 2m - 1} \div \frac{m^3 - 2m^2 - 15m}{m^2 + 3m}$.
20. $\frac{ac + ad + bc + bd}{ad + bd - ac - bc} \div \frac{2ac + 2bd + 2bc + 2ad}{a + b}$.
21. $\frac{m^2 - n^2}{m^2 + n^2} \times \frac{m^4 - n^4}{m^4 - n^4} \div \frac{m^2 + n^2}{n^2 - m^2}$.
22. $\frac{b^3 - c^3}{c^3 - b^3} \div \left(\frac{b + c}{b^2 - 2bc + c^2} \times \frac{b^2 - c^2}{b^3 + c^3}\right)$.

ADDITION AND SUBTRACTION OF FRACTIONS

253. **Method.**—In addition and subtraction of fractions in algebra we follow the same method as in arithmetic.

254. **Least Common Denominator** (L. C. D.).—The L. C. D. of two or more fractions is the L. C. M. of all the denominators of the fractions. We can add and subtract only such fractions as have the same denominators.

(*a*) In arithmetic $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

(*b*) In algebra $\frac{x}{b} + \frac{y}{b} = \frac{x + y}{b}$. Here we have the same denominator, namely b.

(*c*) Find the value of $\frac{x}{a} + \frac{y}{b}$.

Solution. 1. The L. C. D. $= ab$.

2. Change each fraction to an equivalent fraction having **ab** as a denominator as follows:

$\frac{ab}{a} = b$. Multiplying both numerator and denominator of $\frac{x}{a}$ by b, we have $\frac{x}{a} \times \frac{b}{b} = \frac{bx}{ab}$.

In like manner, $\frac{y}{b} = \frac{ay}{ab}$.

$$\therefore \frac{x}{a} + \frac{y}{b} = \frac{bx}{ab} + \frac{ay}{ab} = \frac{bx + ay}{ab}. \qquad \text{From 2.}$$

Therefore, to **add** and **subtract** fractions:

1. *Find the L. C. D. of all the denominators.*

2. *Reduce the given fractions to equivalent fractions having the same denominator.*

3. *Write the algebraic sum of the numerators over the L. C. D.*

4. *Collect like terms in the numerator, if any occur.*

5. *Reduce to lower terms if possible.*

PROBLEMS

Perform the indicated operations: Check as in §31.

1. $\frac{3}{4} + \frac{5}{6}$. 2. $\frac{5}{7} + \frac{2}{3}$. 3. $\frac{3}{6} - \frac{7}{15}$.

4. $\frac{a}{6} - \frac{b}{15}$. 5. $\frac{5a}{2} + \frac{3a}{6}$. 6. $\frac{3x}{7} + \frac{5x}{3}$.

7. $\frac{3}{a} - \frac{2}{b}$. 8. $\frac{7}{x} + \frac{5}{y}$. 9. $\frac{2}{m} + \frac{3}{n} - \frac{7}{p}$.

10. $\frac{2a}{5} + \frac{3a}{8} - \frac{a}{2}$. 11. $\frac{8x}{y} - \frac{3y}{x}$. 12. $\frac{7a}{b} - \frac{5b}{a}$.

13. $\frac{3x + 4}{12} - \frac{2x + 5}{18}$.

Solution 1.—The L. C. D. = 36.

§252–2. $\frac{\quad}{36} - \frac{\quad}{36}$,

$\frac{3(3x + 4)}{36} - \frac{2(2x + 5)}{36}$,

$\frac{36}{12} = 3$, then $3(3x + 4)$.

$\frac{36}{18} = 2$, then $2(2x + 5)$.

§252–3. $\frac{(9x + 12) - (4x + 10)}{36}$.

$\frac{9x + 12 - 4x - 10}{36}$,

Note that $(4x + 10)$ is preceded by the minus sign. When the parentheses are removed, we write $-4x - 10$.

§252–4. $\frac{5x + 2}{36}$. *Ans.*

14. $\frac{2x - 4}{3} + \frac{3x + 5}{2}$. 15. $\frac{4r + 5}{5} - \frac{3r - 2}{3}$. 16. $\frac{3m - 7}{6} - \frac{2m + 5}{5}$.

17. $\frac{2b + a}{7} + \frac{3b - 2a}{14}$. 18. $\frac{3c - 2d}{21} - \frac{4c + 5d}{14}$. 19. $\frac{3x - 5y}{35} - \frac{2x + 7}{15}$.

20. $\frac{3r + 2s}{27} - \frac{4r + s}{81}$. 21. $\frac{2k + 5}{17} + \frac{3k - 7}{68}$.

22. $\frac{2x - 3y}{15} - \frac{x + 2y}{25} + \frac{y + x}{30}$. 23. $\frac{3x - 5}{16} + \frac{2x + 6}{48} - \frac{x - 9}{32}$.

24. $\frac{3r + s}{9} + \frac{2r - 5s}{27} + \frac{6r + s}{18}$. 25. $\frac{5m + 2n}{17} - \frac{13m - n}{34} - \frac{n - 2m}{51}$.

26. $\frac{3}{x} + \frac{2}{y} - \frac{5}{z}$. 27. $\frac{4}{ab} - \frac{6}{bc}$.

28. $\frac{15}{rs} + \frac{12}{st}$. 29. $\frac{a}{am} - \frac{b}{4m^2}$.

30. $\frac{a}{am^2} - \frac{b}{mn} - \frac{c}{3n^2}$. 31. $\frac{m - n}{xy} - \frac{n}{yz} + \frac{m + n}{xz}$.

32. $\frac{3k}{km} - \frac{2l}{lm} + \frac{5m}{mk}$. 33. $\frac{x - y}{xy} + \frac{y - z}{xz} + \frac{z - x}{yz}$.

34. $\frac{c+d}{cd}+\frac{d-e}{de}-\frac{e+c}{ce}.$ 35. $\frac{2a+b}{ab}-\frac{2c-b}{bc}-\frac{b-c}{ca}.$

36. $\frac{2}{a^2+ab}-\frac{3}{a^2-ab}.$

Solution.—See prob. 13.

1. Factor the denominators. $\frac{2}{a(a+b)}-\frac{3}{a(a-b)}.$

$\therefore$ The L. C. D. is $a(a+b)(a-b)$.

2. $=\frac{2(a-b)}{a(a+b)(a-b)}-\frac{3(a+b)}{a(a+b)(a-b)}$

3. $=\frac{2(a-b)-3(a+b)}{a(a+b)(a-b)}=\frac{2a-2b-3a-3b}{a(a+b)(a-b)}$

4. $=\frac{-a-5b}{a(a^2-b^2)}=\frac{-(a+5b)}{a(a^2-b^2)}$ or $-\frac{a+5b}{a(a^2-b^2)}.$ Result.

37. $\frac{b}{a+b}+\frac{a}{a-b}-\frac{ab}{a^2-b^2}.$ The L. C. D. $=(a+b)(a-b)$.

$$=\frac{b(a-b)}{(a+b)(a-b)}+\frac{a(a+b)}{(a+b)(a-b)}-\frac{ab}{(a+b)(a-b)}$$

$=\frac{ba-b^2+a^2+ab-ab}{(a+b)(a-b)}=\frac{a^2-b^2+ab}{(a+b)(a-b)}.$ Result.

38. $\frac{x}{x+1}-\frac{1}{x-1}.$ 39. $\frac{4}{3m-2}-\frac{3}{m+1}.$

40. $\frac{x-2}{x+2}-\frac{2+x}{x-2}.$ 41. $\frac{y+3}{y-3}-\frac{y-3}{y+3}-\frac{36}{y^2-9}.$

42. $\frac{1}{2x+2y}-\frac{1}{3x+3y}.$ 43. $\frac{3x}{y-x}-\frac{2y}{y+x}.$

44. $\frac{4a}{a-3}+\frac{5a}{a+3}-\frac{6a}{a^2-9}.$ 45. $\frac{m+n}{m-n}+\frac{m-n}{m+n}-\frac{2mn}{m^2-n^2}.$

46. $\frac{3x+1}{2x+1}+\frac{2x+1}{2x-1}-\frac{4x}{4x^2-1}.$ 47. $\frac{ax}{a-x}+\frac{a}{a^2+ax+x^2}-\frac{x}{a^3-x^3}.$

48. $\frac{3}{y}+\frac{5}{y-3}+\frac{7}{y^2-6y+9}.$ 49. $\frac{1}{b+c}-\frac{1}{a+c}-\frac{1}{a+b}.$

50. $\frac{2}{x^2-4}-\frac{1}{2-x}.$ **Hint.** $-\frac{1}{2-x}=\frac{1}{x-2}.$ §247–II*a*.

51. $\frac{a-1}{a^2-3a+2}+\frac{a+2}{a^2-5a+6}-\frac{a-3}{a^2-4a+3}.$

Hint. *The L. C. D. is* $(a-1)(a-2)(a-3)$.

$$\therefore \frac{(a-3)(a-1)}{(a-1)(a-2)(a-3)}+\frac{(a-1)(a+2)}{(a-1)(a-2)(a-3)}-\frac{(a-2)(a-3)}{(a-1)(a-2)(a-3)}$$

$$=\frac{a^2-4a+3+a^2+a-2-a^2+5a-6}{(a-1)(a-2)(a-3)}=\frac{a^2+2a-5}{(a-1)(a-2)(a-3)}.$$

Result.

52. $\frac{1}{m}+\frac{m+2}{m^2-6m+9}+\frac{m+3}{m^2-9}.$

53. $\frac{4}{x^2-3x+2}-\frac{5}{x^2-x-6}+\frac{6}{4-x^2}.$

54. $\frac{4}{2x^2-x-3}-\frac{3}{2x^2+x-3}-\frac{6}{2x^2-5x+3}.$

55. $a+\frac{1}{a-b}$ Solution. $\frac{a}{1}+\frac{1}{a-b}=\frac{a^2-ab+1}{a-b}.$ *Ans.* Or use §256.

56. $a-3+\frac{4a-5}{2a}.$ 57. $13x+\frac{5}{7x}-\frac{1}{2}.$

58. $\frac{a-b}{a+b}+3.$ 59. $4-\frac{3x+y}{2y+x}.$

60. $2a - 1 - \frac{3a+2}{2a-3}$.

61. $\frac{5x}{x^2-4} - \frac{3}{2-x}$.

62. $\frac{1}{r^2-rs} - \frac{2}{s^2-rs}$.

63. $\frac{a}{3-a} + \frac{2}{a-3} - 1$.

64. $\frac{m}{m+2} - \frac{m}{2-m} + 3$.

65. $\frac{1}{x+1} - \frac{x}{x^2-x+1} + \frac{x+2}{x^3+1}$.

66. $\frac{m}{m^2-2mn+n^2} + \frac{n}{m^2-n^2}$.

67. $m^2 - mn + n^2 + \frac{2m^3}{m+n}$.

68. $\frac{2x}{x-1} + 2 - \frac{3x}{x-2}$.

255. Mixed Expressions.—A mixed expression is one which is composed of both integral and fractional parts.

Thus, $2 + \frac{1}{3}$, $a + \frac{b}{y}$, and $2 - \frac{b}{c}$ are mixed expressions.

256. To Change a Mixed Expression to a Fraction.

In arithmetic, $2 + \frac{1}{3} = \frac{7}{3}$. For $2 = \frac{6}{3}$, and $\frac{6}{3} + \frac{1}{3} = \frac{7}{3}$.

Again, $3 - \frac{1}{2} = \frac{6}{2} - \frac{1}{2}$ or $\frac{5}{2}$.

In algebra, $a + \frac{b}{y} = \frac{ay}{y} + \frac{b}{y}$ or $\frac{ay+b}{y}$, and $b - c - \frac{d}{x} = \frac{bx}{x} - \frac{cx}{x} - \frac{d}{x}$ or $\frac{bx-cx-d}{x}$.

Therefore, to change a mixed expression to a fraction, multiply the integral part of the expression by the denominator of the fractional part and add or subtract the numerator of the fraction as indicated by the sign before the fraction.

Write this result over the denominator.

Thus, $a + b - \frac{4ab-b^2}{a-b} = \frac{(a+b)(a-b)-(4ab-b^2)}{a-b} = \frac{a^2-4ab}{a-b}$.

PROBLEMS

Change the following to fractions: Check as in §31.

1. $2a + \frac{4a}{5}$.

2. $3p + \frac{5p}{6}$.

3. $2x - \frac{3x}{5}$.

4. $5x - \frac{3k}{7}$.

5. $m + \frac{m-2}{3}$.

6. $3r - \frac{2r+5}{7}$.

7. $2m + \frac{3n-2}{3}$.

8. $6c - \frac{3m+2}{9}$.

9. $5s - \frac{2+t}{7}$.

10. $7c + \frac{3a+b}{5}$.

11. $3r + \frac{5a-r}{3rs}$.

12. $5k - \frac{3p+q}{kq}$.

13. $a + 2 + \frac{3ac-a}{c}$.

14. $b + c - \frac{3bc+c^2}{b-c}$.

15. $x + y - \frac{x^2-y^2}{x-y}$.

16. $2r - 3 + \frac{4r^2-9}{2r-3}$.

17. $a + b - \frac{3ab-2b^2}{a-b}$.

18. $r + s - \frac{r^3-s^3}{r^2-rs+s^2}$.

19. $m + n + \frac{3mn+n^2}{m-2n}$.

20. $y - 2 - \frac{y^2-6}{y+2}$.

257. To Change a Fraction to an Integral or Mixed Expression. In arithmetic, $\frac{8}{3}$ is made up of $\frac{6}{3}+\frac{2}{3}$ or $2+\frac{2}{3}$.

In algebra, $\dfrac{ab+c}{a}=\dfrac{ab}{a}+\dfrac{c}{a}$ or $b+\dfrac{c}{a}$,

Also $\dfrac{mn-r}{n}=\dfrac{mn}{n}-\dfrac{r}{n}$ or $m-\dfrac{r}{n}$.

Therefore to change an improper fraction to an integral or mixed expression, divide the numerator by the denominator. Write the remainder, if any, over the denominator and add this fraction to the integral quotient.

Thus, $\dfrac{a^2+4ab+3b^2}{a+2b}=a+2b+\dfrac{-b^2}{a+2b}$ or $a+2b-\dfrac{b^2}{a+2b}$.

PROBLEMS

Change the following to integral or mixed expressions:

1. $\dfrac{x^2-4}{x-2}$. 2. $\dfrac{a^2-2ab+b^2}{a-b}$. 3. $\dfrac{4x^2-x-1}{x}$.
4. $\dfrac{a^2-2ab+5b^2}{a+3b}$. 5. $\dfrac{3x^2-4xy+y^2}{x-y}$. 6. $\dfrac{ab+ac}{b+c}$.
7. $\dfrac{2x^2-y^2}{2x}$. 8. $\dfrac{4y^2-2x^2}{2(y-x)}$. 9. $\dfrac{a^3+b^3}{a+b}$.
10. $\dfrac{a^3+b^3}{a-b}$. 11. $\dfrac{5a^2-2ab+3}{2a}$. 12. $\dfrac{3x^2-2xy+y^2}{x-y}$.
13. $\dfrac{5x^3+3x^2-2x+1}{x^2-1}$. 14. $\dfrac{4x^2-ax-1}{4x^2+x-2}$.

258. **Multiplication and Division of Mixed Expressions.**—In the following problems simplify each mixed expression before performing the indicated operation.

PROBLEMS

1. $(4+\frac{2}{3})(3-\frac{1}{2})$. 2. $(3-\frac{2}{5})\div(4+\frac{1}{3})$.
3. $\left(1+\dfrac{a}{b}\right)\left(\dfrac{1}{a+b}\right)$. 4. $\left(m+\dfrac{n^2}{m}\right)\left(1-\dfrac{m^2}{m^2+n^2}\right)$.
5. $\left(\dfrac{a}{x}-\dfrac{c}{b}\right)\div\left(\dfrac{a}{x}+\dfrac{c}{b}\right)$. 6. $\left(4-\dfrac{1}{b}\right)\left(\dfrac{b}{4b-1}\right)$.
7. $\left(\dfrac{b^2}{b^2-a^2}\right)\left(1+\dfrac{b}{a}\right)$. 8. $\left(4-\dfrac{y}{y-x}\right)\div\left(4+\dfrac{y}{y-x}\right)$.
9. $\left(a+2-\dfrac{3}{a}\right)\div\left(a-5+\dfrac{4}{a}\right)$. 10. $\left(x+y+\dfrac{x^2}{y}\right)\left(\dfrac{y^2-yx}{x^3+x^2y+xy^2}\right)$.

259. **Complex Fractions.**—A complex fraction is a fraction one or both of whose terms are fractions or mixed expressions.

Thus,

1. $\dfrac{x}{\dfrac{b}{a}}$, 2. $\dfrac{m}{1+\dfrac{m}{n}}$, 3. $\dfrac{\dfrac{x}{x-y}}{\dfrac{y}{x-y}}$, and 4. $\dfrac{\dfrac{a}{a-2}-\dfrac{a}{a+2}}{\dfrac{a}{a-2}+\dfrac{a}{a+2}}$,

are complex fractions.

Follow the solutions of the above illustrations.

1. $\frac{x}{\frac{b}{a}} = \frac{x}{1} \times \frac{a}{b} = \frac{ax}{b}$. *Ans.*

2. $\frac{m}{1+\frac{m}{n}} = \frac{m}{\frac{n+m}{n}} = \frac{m}{1} \times \frac{n}{m+n} = \frac{mn}{m+n}$. *Ans.*

3. $\frac{\frac{x}{x-y}}{\frac{y}{x-y}} = \frac{x}{\cancel{x-y}} \times \frac{\cancel{x-y}}{y} = \frac{x}{y}$. *Ans.*

4. $$\frac{\frac{a}{a-2}-\frac{a}{a+2}}{\frac{a}{a-2}+\frac{a}{a+2}} = \frac{\frac{a^2+2a-a^2+2a}{(a-2)(a+2)}}{\frac{a^2+2a+a^2-2a}{(a-2)(a+2)}} = \frac{\frac{4a}{(a-2)(a+2)}}{\frac{2a^2}{(a-2)(a+2)}},$$

$$= \frac{\overset{2}{\cancel{4a}}}{\cancel{(a-2)}\cancel{(a+2)}} \times \frac{\cancel{(a-2)}\cancel{(a+2)}}{\underset{a}{\cancel{2a^2}}} = \frac{2}{a}. \quad \textit{Ans.}$$

Simplify the following:

1. $\frac{\frac{1}{2}-\frac{2}{3}}{\frac{1}{2}+\frac{2}{3}}$.

[illegible] $\frac{1+\frac{1}{5}}{3+\frac{3}{5}}$.

3. $\frac{\frac{3xy}{9x^2y}}{y}$.

4. $\frac{1+\frac{1}{2x}}{x-\frac{1}{4x}}$.

5. $\frac{x-\frac{1}{x^2}}{1-\frac{1}{x}}$.

6. $\frac{\frac{4}{x}}{2+\frac{1}{x}}$.

[illegible] $\frac{\frac{x}{y}-\frac{a}{b}}{\frac{x}{y}+\frac{a}{b}}$.

8. $\frac{x-2+\frac{1}{x}}{x-3+\frac{2}{x}}$.

9. $\frac{x+4-\frac{21}{x}}{x-4+\frac{3}{x}}$.

10. $\frac{x-\frac{y^2}{x+y}}{x+\frac{y^2}{x+y}}$.

11. $\frac{a-1+\frac{a^2+2a+1}{a-1}}{a+1+\frac{a^2-2a+1}{a+1}}$.

12. $\frac{\frac{m}{m+n}}{1+\frac{m}{m-n}}$.

[illegible] $\frac{\frac{a}{a+b}-\frac{b}{b-a}}{\frac{a}{a+b}+\frac{b}{a-b}}$.

14. $\frac{\frac{1}{x}\times\frac{1}{y}}{\frac{x}{y}\times\frac{y}{1}} \div \frac{1}{x^2y}$.

[illegible] $\frac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}} \div \frac{a-b}{a+b}$.

16. $\left(1-\frac{x}{y}\right)\frac{y}{x-y} \div \frac{y^2-x^2}{x^2-y^2}$.

17. $\frac{\frac{m^2-mn+1}{mn}}{\frac{m^2-mn-1}{mn}} \div \frac{m^2-2mn+n^2-1}{m-n-1}$.

CHAPTER VII

FRACTIONAL AND LITERAL EQUATIONS

260. **Fractional Equations.**—A fractional equation is an equation containing one or more fractions.

The fractional equation is the tool without which the engineer is unable to make a large number of his computations. Thus in planning the construction of a steam engine, the engineer knows that the formula, or fractional equation, $HP = \frac{PLAN}{33{,}000}$, enables him to compute all the dimensions of the cylinder which will develop a certain horsepower.

Note.—*An equation, none of whose terms contain fractions, is usually called an integral equation.*

261. **Method of Solving Fractional Equations.**

First, clear the equation of fractions by multiplying both members of the equation by the L. C. D.

Second, solve the resulting simple equation.

Note.—*If a fraction is preceded by a minus sign, the signs of all the terms in the numerator are changed when the fraction is eliminated.*

Illustrations.—1. Solve for x, $\frac{x}{3} - \frac{x-1}{8} = x - 7$.

Solution.—Multiplying each term of both members of the equation by 24, the L. C. D., thus, $24\left(\frac{x}{3}\right) - 24\left(\frac{x-1}{8}\right) = 24\left(\frac{x-7}{1}\right)$.

Simplifying, $8x - 3x + 3 = 24x - 168$.
Transposing, $8x - 3x - 24x = -3 - 168$.
Collecting like terms, $-19x = -171$.
Dividing by -19, $x = 9$.

Check.
$$\frac{9}{3} - \frac{9-1}{8} = 9 - 7.$$
$$3 - 1 = 2.$$
$$2 \equiv 2.$$

2. Solve for x, $\frac{3x+1}{3x-1} - \frac{3x-1}{3x+1} = \frac{12}{9x^2-1}$.

Solution.—The L. C. D. is $(3x + 1)(3x - 1)$.

Multiplying each term of both members of the equation by the L. C. D., we have, $(3x + 1)^2 - (3x - 1)^2 = 12$.

Thus, $\frac{3x+1}{\cancel{3x-1}} \times (3x + 1)(\cancel{3x-1}) = (3x + 1)^2$, etc.

Expanding, $9x^2 + 6x + 1 - 9x^2 + 6x - 1 = 12$.
Collecting like terms, $12x = 12$.
Dividing by 12, $x = 1$.

PROBLEMS

Solve and check:

1. $\frac{x}{2} = 3.$

2. $\frac{2x}{3} - \frac{2}{3} = \frac{2x}{5} - \frac{2}{5}.$

3. $\frac{2x}{5} - \frac{x}{2} = \frac{x}{4} - \frac{7}{2}.$

4. $\frac{x}{3} - \frac{3x}{5} + \frac{7x}{5} = \frac{17}{3}.$

5. $\frac{2x}{3} - \frac{2x+1}{4} = \frac{1}{4}.$

6. $\frac{x+1}{2} + \frac{x+3}{3} = \frac{29}{6}.$

7. $\frac{3x-1}{2} - \frac{2x+1}{3} - \frac{4x-5}{4} = \frac{1}{4}.$

8. $\frac{2x-3}{3} + \frac{x+1}{4} = \frac{5x-2}{5}.$

9. $5 = \frac{1}{2}(2x-4).$

10. $\frac{2}{3}(x-5) - \frac{4}{5}(3x-6) + \frac{134}{15} = 0.$

11. $\frac{3-2x}{8} - \frac{x-3}{6} + 2 = \frac{4-x}{2} + \frac{19}{24}.$

12. $\frac{3x}{5} - \frac{1}{4}(x+3) = \frac{x+4}{2} - \frac{7}{2}.$

13. $\frac{6}{x+1} - \frac{1}{x-1} = \frac{3}{x^2-1}.$

14. $\frac{x+3}{x-2} - \frac{29}{x^2-4} = \frac{x-2}{x+2}.$

15. $\frac{3}{2x-1} + \frac{5}{4x^2-1} - \frac{14}{2x+1} = 0.$

16. $\frac{2x-3}{2x-4} + \frac{1}{4} = \frac{x+5}{3x-6}.$

17. $\frac{x}{3} - \frac{x^2-5x}{3x-7} = \frac{16}{11}.$

18. $\frac{4}{x+2} + \frac{7}{x+3} - \frac{37}{x^2+5x+6} = 0.$

19. $\frac{2}{3}(x-1) = \frac{1}{2}(x-2).$

20. $3\left(\frac{2x}{3} - \frac{1}{2}\right)\left(\frac{x}{2} + \frac{2}{3}\right) = x^2.$

21. $\frac{3}{3-x} + \frac{6}{3+x} = \frac{8x+5}{9-x^2}.$

22. $\frac{x^2-x+1}{x-1} = \frac{x^2+x+1}{x+1} + \frac{x}{x^2-1}.$

23. $\frac{2x-3}{3x-2} = \frac{2x+3}{3x+2}.$

24. $\frac{x+2}{x-2} = \frac{x-2}{x+2}.$

262. Literal Equations.—A literal equation is an equation containing one or more known literal expressions.

Illustrations.—1. Solve for x, $\frac{a+b}{x} = \frac{1}{x+1}.$

Solution.—Multiplying both members of the equation by the L. C. D., $x(x+1)$,

we have, $(a+b)(x+1) = x.$

Expanding, $ax + a + bx + b = x.$

Transposing, $ax + bx - x = -a - b.$

Factor the left member of the equation to get the coefficient of x. Then, $(a+b-1)x = -a-b.$

Dividing by the coefficient of x, $x = \frac{-a-b}{a+b-1} = \frac{a+b}{1-a-b}.$

The value of x contains the literal numbers **a** and **b** because they are known numbers.

2. Solve for x, $\frac{ax}{3a+b} + \frac{bx}{3a-b} = \frac{3a^2x+b^2}{9a^2-b^2}.$

Solution.—Multiplying both members of the equation by the L. C. D., $9a^2 - b^2$,

we have, $ax(3a-b) + bx(3a+b) = 3a^2x + b^2.$

Expanding, $3a^2x - abx + 3abx + b^2x = 3a^2x + b^2.$

Transposing, $3a^2x - abx + 3abx + b^2x - 3a^2x = b^2.$

Collecting like terms, $2abx + b^2x = b^2.$

Factoring, $(2ab + b^2)x = b^2.$

Dividing by the coefficient of x, $x = \frac{b^2}{2ab+b^2} = \frac{b}{2a+b}.$

PROBLEMS

Solve for x. Check.

1. $\frac{x}{a} = 5.$ 2. $-\frac{x}{a} = b.$ 3. $ax - b = 0.$

4. $\frac{2}{x} = \frac{1}{3a}.$ 5. $cx - dx = e.$ 6. $\frac{ax}{d} = \frac{b}{c}.$

7. $a = -\frac{x}{b}.$ 8. $a = -bx.$ 9. $2ax = 3b - 5cx.$

10. $\frac{m+n}{x} = \frac{a+b}{2}.$ 11. $\frac{x}{a} + \frac{x}{3a} = 4.$ 12. $\frac{x+a}{x-b} = \frac{3}{4}.$

13. $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$ 14. $\frac{c+1}{c-1} = \frac{c+x}{c-x}.$

15. $\frac{1}{2}\left(\frac{x}{m} + 1\right) = \frac{1}{3}\left(\frac{x}{m} - 1\right).$

16. $\frac{x+a}{3} = \frac{x-a}{2}.$ 17. $\frac{x}{a} = \frac{x+a}{3a}.$ 18. $\frac{1}{a} - \frac{1}{x} = \frac{1}{x} - \frac{1}{b}.$

19. $\frac{2}{3}\left(\frac{x}{a} + 1\right) = \frac{3}{4}\left(\frac{x}{2} - 1\right).$ 20. $\frac{a}{x} = c(a - b) + \frac{b}{x}.$

21. $\frac{9a}{b} - \frac{3x}{b} = \frac{4b}{a} - \frac{2x}{a}.$ 22. $\frac{x-m}{n-x} = \frac{x-n}{m-x}.$ 23. $\frac{x-b}{2} = \frac{(x-a)^2}{2x-b}.$

24. $\frac{x}{a} - \frac{x}{b} - \frac{x}{c} = 1.$ 25. $\frac{3cx}{c-d} - 2c = 5x.$ 26. $\frac{m+nx}{m+n} = \frac{c+dx}{c+d}.$

27. $\frac{x}{c} + \frac{x}{c-d} = \frac{c}{c+d}.$ 28. $\frac{a+2x}{a-2x} = -\frac{4x^2}{4x^2-a^2}.$

29. $$\frac{a - \dfrac{x}{a-b}}{a + \dfrac{x}{a-b}} - 1 = \frac{a}{b}.$$

30. How many sides has an equiangular polygon if each angle is 120°?

31. How many sides has an equiangular polygon if each angle is $152\frac{4}{13}$°?

32. Can the following be interior angles of an equiangular polygon? 150°; 108°; 140°; 155°; 85°; 95°; 120°; 160°. If so, compute the number of sides.

33. Can the following be exterior angles of an equiangular polygon? 120°; 60°; 30°; 10°; 18°. If so, compute the number of sides.

34. How many sides has a polygon, if the sum of the interior angles equals the sum of the exterior angles?

35. How many sides has a polygon, if the sum of the interior angles is double the sum of the exterior angles?

36. How many sides has a polygon, if the sum of the exterior angles is double the sum of the interior angles?

37. Each exterior angle of an equiangular polygon is equal to an angle of an equilateral triangle. Find the number of sides in the polygon.

38. The acute angles of a right triangle are to each other as $\frac{3}{4}$ is to $\frac{5}{8}$. Find them.

39. The area of a triangle is expressed by the formula $A = \frac{1}{2}bh$. (*a*) Find A, when $b = 10$ and $h = 7.5$. (*b*) Find b, when $A = 98$ and $h = 12$. (*c*) Solve for b. (*d*) Solve for h.

40. The area of a trapezoid is expressed by the formula $A = \frac{1}{2}h(b + b_1)$. (*a*) Find A, when $h = 8$, $b = 12.5$, and $b_1 = 9.5$. (*b*) Find b, when $A = 180$, $h = 9$, and $b_1 = 12$. (*c*) Solve for h. (*d*) Solve for b_1.

41. The area of a circle is expressed by the formula $A = \pi r^2$. $\left(\pi = \frac{22}{7}\right)$. (*a*) Find r, when $A = 440$. (*b*) Solve for r^2.

42. $\frac{1}{f} = \frac{1}{a} + \frac{1}{b}$ is a formula used in physics. (*a*) Find f, when $a = 3$ and $b = 5$. (*b*) Solve for a. (*c*) Solve for f.

43. The volume of a rectangular parallelopiped is expressed by the formula $V = abc$. (*a*) Find a, when $V = 300$, $b = 5$, and $c = 6$. (*b*) Solve for a. (*c*) Solve for b. (*d*) Solve for c.

44. The lateral area of a regular pyramid is expressed by the formula $S = \frac{1}{2}P \times L$. (*a*) Find S, when $P = 15.5$ and $L = 12$. (*b*) Find P, when $S = 30$ and $L = 6$. (*c*) Solve for L. (*d*) Solve for P.

45. The lateral area of a cone of revolution is expressed by the formula $S = \pi RL$. (*a*) Find R, when $S = 425$ and $L = 28$. (*b*) Solve for R. (*c*) Solve for L.

46. The formula for velocity is $V = gt$. (*a*) Find t, when $V = 128$ and $g = 32$. (*b*) Solve for g. (*c*) Solve for t.

47. $h = \frac{WV^2}{1.25r}$ is a formula used in engineering. (*a*) Solve for W. (*b*) Solve for r. (*c*) Solve for V^2.

48. The formula for the amount of money on interest at a given rate after a given time is $A = P + \frac{Prt}{100}$. (*a*) Find A, when $P = \$1000$, $r = 6$, and $t = 5$ years. (*b*) Find t, when $A = \$900$, $P = \$750$, and $r = 5$. *Ans.* $t = 4$. (*c*) Solve for P. (*d*) Solve for r. (*e*) Solve for t.

49. $I = \frac{E}{R}$ is a formula used in electrical work. (*a*) Find I, when $E = 300$ and $R = 400$. (*b*) Find E, when $I = .5$ and $R = 250$. (*c*) Solve for R. (*d*) Solve for E.

50. The formula for converting a temperature of C degrees Centigrade into its equivalent of F degrees Fahrenheit is $F = \frac{9}{5}C + 32$. (*a*) Find F, when $C = 25$. (*b*) Find C, when $F = -40$. (*c*) Solve for C. *Ans.* $C = \frac{5}{9}(F - 32)$.

51. The pressure per square inch allowed by the U. S. Steamboat Inspection rules on boilers is given by the formula, $P = \frac{T \times t}{6 \times R}$. P = pressure in pounds per sq. in. T = tensile strength of the material of the shell (pounds per sq. in.). t = thickness of plate in inches. R = radius of shell in inches. If $T = 60{,}000$ pounds per sq. in., which boiler will stand a greater pressure; one 60″ in diameter, ½″ thickness of plate or one 48″ in diameter, $\frac{7}{16}$″ thickness of plate?

52. A boiler is to stand a pressure of 160 pounds per sq. in. Find the radius if $T = 50{,}000$ and $t = \frac{3}{8}$″. Use the formula in problem 51.

53. The indicated horsepower of an engine is expressed by the formula,

$$HP = \frac{PLAN}{33{,}000}.$$

HP = horsepower.
P = average cylinder pressure in lbs. per sq. inch.
A = area of piston in sq. inches.
L = length of stroke in feet.
N = Strokes per minute, or 2 × r.p.m. of drivers or flywheel for each cylinder.

Suppose we wish to find, from the above formula, the diameter of the cylinder in terms of the other letters in the formula. From the formula we get $33{,}000\,HP = PLAN$. Divide both sides by PLN, $\frac{33{,}000HP}{PLN} = A$. But $A = \frac{\pi d^2}{4} = .7854d^2$. Substituting the value of A we get $.7854d^2 = \frac{33{,}000HP}{PLN}$.

$$\therefore d^2 = \frac{33{,}000HP}{.7854\,PLN}.$$

Taking square root, $d = \sqrt{\frac{33{,}000HP}{.7854\,PLN}}$. (*a*) Solve for P. (*b*) Solve for L. (*c*) Solve for A. (*d*) Solve for N.

54. From the preceding formula find the value of N, when $P = 175$ lbs. per square inch, the cylinder 24″ × 2⅓′ and $HP = 1250$.

55. Using the same formula, find the pressure in lbs. per sq. in. upon the cylinder if $HP = 1000$. $N = 160$, $A = 530.93$, and L = 2 feet 6 inches.

56. A double acting steam engine develops 1100 *HP* with an average steam pressure of 150 lb. per sq. in., the length of the stroke is 2 ft., and the engine runs at a speed of 200 r.p.m. (400 strokes). What is the diameter of the cylinder?

57. The Philadelphia Column Formula is

$$S = \frac{P}{A} = \frac{16,250}{1 + \dfrac{l^2}{11,000r^2}}$$

The Straight Line Column Formula, used by New York, Chicago, St. Louis, and The American Railway Engineers Association is

$$S = \frac{P}{A} = 16,000 - 70\frac{l}{r}, \text{ where}$$

S = unit stress (pounds per sq. in.).
P = total load in pounds.
A = area of cross section in sq. in.
l = length in inches.
r = least radius of gyration in inches.

Find the total load which a column will support, if $A = 11\frac{1}{2}$ sq. in., $l = 12$ ft., and $r = 2$ inches.

Solve by both formulas. Compare results.

58. Find the area required and the unit stress of a column supporting a total load of 100 short tons, if the unsupported length is 16 ft. and the least radius of gyration is 1.80 in. Solve by both formulas.

59. The formula for coefficient of Elasticity is $E = \frac{Pl}{AD}$. (*a*), Solve for D. (*b*), Solve for A. (*c*), Solve for P.

60. Flexural Formula for Beams.

$$S = \frac{MC}{I}. \quad \text{Show that } S = \frac{M}{\left(\dfrac{I}{C}\right)}. \quad \frac{I}{C} \text{ is the Section Modulus.}$$

61. Solve for x.

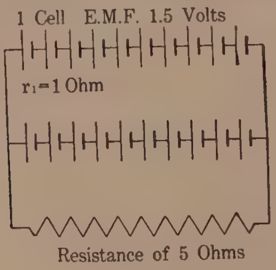
$$\frac{\dfrac{b}{b+1}}{\dfrac{2}{x-1}} = \frac{\dfrac{b}{b-1}}{\dfrac{4}{x+1}}.$$

62. The current expressed in amperes flowing through an electrical circuit may be found by the formula

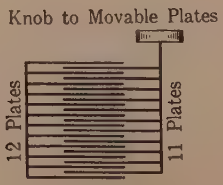
$$I = \frac{ne}{R_e + \dfrac{nr_i}{m}},$$

where I = current in amperes; n = no. of cells in series, e = e.m.f. of 1 cell; m = no. of groups in parallel; R_e = external resistance in ohms; r_i = internal resistance in ohms.

(*a*) What current would 20 dry cells of 1.5 volts e.m.f. supply to an external resistance of 5 ohms, if the internal resistance of each cell is one ohm and they are arranged 10 in series and two groups as shown in drawing? (*b*) If 100 cells were used how much current would they supply?



PROB. 62.



PROB. 63.

63. The capacity of a radio air condenser is 0.001 microfarads. How large is each plate in square inches if they are semicircular in shape, 23 in number, spaced a distance of 0.1 inch apart?

Formula

$$C = \frac{KA(n-1)}{4\pi t \times 9 \times 10^5}$$

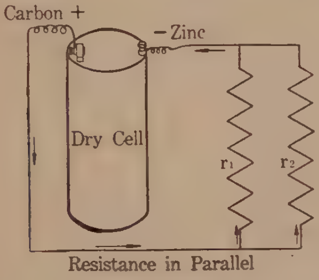
= 20.2 ans

C = capacity in microfarads.
K = constant for air or 1.
A = area of one plate in square centimeters. (1 inch = 2.54 cm.)
n = total number of plates.
t = distance spaced apart.

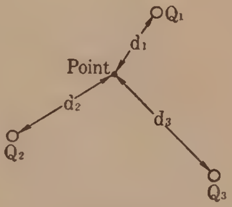
Suppose one plate were used, how large would it be?

64. Two wires are connected in parallel and have a combined resistance of 10 ohms. If one resistance is 12 ohms, find the other.

$R = \dfrac{1}{\frac{1}{r_1} + \frac{1}{r_2}}$, where R = combined resistance in ohms; r_1 = resistance in ohms of first resistance; r_2 = resistance in ohms of second resistance.



PROB. 64.



PROB. 65.

65. Find the potential at a point due to the following charges: one of 40 e.s. units at a distance of 15 cm., another of 50 e.s. units at a distance of 25 cm., and a third of 45 e.s. units at a distance of 30 cm.

$V = \dfrac{Q_1}{d_1} + \dfrac{Q_2}{d_2} + \dfrac{Q_3}{d_3}$; where V = potential in electro-static units. Q_1, Q_2, and Q_3 are the charge in e.s. units, and d_1, d_2, and d_3 are the respective distances of the charges.

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CHAPTER VIII

RATIO, VARIATION, AND PROPORTION

263. **Need of Ratio, Variation, and Proportion.**—In many problems in engineering we deal with related values as used in formulas. It is often necessary to find how a change in the value of one or more quantities effects the values of other quantities in the same problem. That is, we find the variation of one quantity with respect to another. These relations will be brought out more fully by a study of the problems which follow.

264. **Ratio.**—A ratio is a relation between two quantities of the same kind. It is expressed as the quotient of one quantity divided by another. Thus, $\frac{2}{3}$, $\frac{a}{b}$, $\frac{2x}{5x}$, and $\frac{3 \text{ apples}}{5 \text{ apples}}$ are ratios. These ratios may also be written $2:3$, $a:b$, $2x:5x$, and 3 apples:5 apples.

VARIATION

265. **Constant.**—A constant is a number which has the same value throughout the discussion of a problem as 2, a, etc.

266. **Variable.**—A variable (fluxion) is a number which has different values throughout the discussion of the problem.

Illustration.—Given the equation $2x - 3y = 6$. If $x = 6$, then $y = 2$, but if $x = 12$, then $y = 6$. Hence x and y are variables.

267. **Function.**—When two variables, such as **x** and **y**, are so related that a change in the value of **x** causes a corresponding change in the value of **y**, then **y** is said to be a function of **x**. In a similar way **x** may be a function of **y**.

Illustrations.—1. The area of a rectangle is given by the formula $A = bh$. Here A is a function of **b** and **h**.

2. The velocity of a body is given by the formula $v = at$. Here **v** is a function of **t** only, for **a** is a constant.

3. In the equation $y = 2x - 3$, every change in the value of **x** causes a change in the value of **y**. Hence **y** is a function of **x**.

Give numerous other illustrations where one variable depends upon another.

268. **Direct Variation.**—When **x** and **y** are so related that the ratio of **y** to **x** is constant, y varies directly as x. This is written $y \propto x$. It is also written $\frac{y}{x} = K$ or $y = Kx$, where K is a constant.

Illustrations.—1. The number of dollars in the cost **c** of automobiles at a fixed price varies as the number **n**, bought, since the ratio $\frac{c}{n}$ is a constant or the cost of each automobile. $\therefore c \propto n$.

2. In the formula, $s = \frac{1}{2} gt^2$, $\frac{1}{2} g$ is a constant, **s** varies directly as t^2, that is the space passed over varies directly as the square of the time, or $s \propto t^2$.

269. **Inverse Variation.**—When **x** and **y** are so related that the ratio of y to $\frac{1}{x}$ is constant, y varies inversely as x. This is written $y \propto \frac{1}{x}$. It is also written $\frac{y}{\frac{1}{x}} = K$, $xy = K$, or $y = \frac{K}{x}$.

Illustrations.—1. In the formula for the area of a parallelogram, $A = bh$, $b = \frac{A}{h}$. If the area **A** remains constant **b** varies inversely as **h**, because as **h** becomes larger, **b** becomes smaller and vice versa.

2. If **y** varies directly as **x**, and **y** is 200 when **x** is 50, what equation expresses the relation between **y** and **x**?

Solution.—If $y \propto x$, then $\frac{y}{x} = K$ or $y = Kx$. (1)

Substituting $y = 200$ and $x = 50$, we have $200 = 50K$. (2)

$\therefore K = 4$. Substituting the value of K in (1), we have $y = 4x$.

3. The distance a body falls from rest varies as the square of the time it is falling. If a body falls through 144 feet in 3 seconds, how far will it fall in 6 seconds?

Solution.—Let s = distance and t = time.

Then $\frac{s}{t^2} = K$.

Substituting the values of s and t we have, $\frac{144}{9} = K$. Hence, $K = 16$.

$\therefore \frac{s}{36} = 16$, or $s = 576$ feet.

PROBLEMS

1. If **x** varies as **y**, and if $y = 6$ when $x = 10$, find x when $y = 10$.

2. If W varies inversely as P, and W is 8 when P is 30, find **W** when P is 12.

3. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 2 feet is 12.5664 square feet. Find the area of a circle whose radius is 12 feet.

4. The volume of a sphere varies as the cube of its radius and the volume of a sphere whose radius is 1 foot is 4.1888 cubic feet. Find the volume of a sphere whose radius is 3 feet.

5. If **x** varies inversely as **y** and **y** varies inversely as **z**, show that **x** varies directly as **z**.

6. If **x** varies inversely as $y^2 - 1$, and is equal to 24 when $y = 10$, find x when $y = 5$.

7. If $5x - y$ varies as $10x - 11y$, and when $x = 7$, $y = 5$, find the equation showing the relation between x and y. *Ans.* $5x - 7y = 0$.

8. If the square root of a varies as the cube root of b, and if $a = 4$ when $b = 8$, find the equation between a and b.

9. The distance of the offing at sea varies as the square root of the height of the eye above the sea level. The distance is 3 miles when the height is 16 feet. Find the distance when the height is 64 feet. *Ans.* 6 miles.

10. Given the formula $b = \frac{c}{d}$. How does b vary when c becomes larger? When c becomes smaller? When d becomes larger? When d becomes smaller?

11. Given the formula $\frac{I}{C} = \frac{M}{S}$. Does $\frac{I}{C}$ become larger or smaller when M becomes larger? When M becomes smaller? When S becomes larger? When S becomes smaller?

12. Given the formula $\frac{a}{c} = \frac{d}{e}$. How does $\frac{d}{e}$ vary when a becomes smaller? When c becomes smaller?

13. Given the formula $P = \frac{b_1 b_2}{c_1 + c_2}$. b_1, b_2, c_1, and c_2 are all positive.

Answer larger or smaller to the following.

(*a*) If b_1 becomes larger, P will become ——————.
(*b*) If b_2 becomes larger, P will become ——————.
(*c*) If c_1 becomes larger, P will become ——————.
(*d*) If c_2 becomes larger, P will become ——————.
(*e*) If b_1 becomes smaller, P will become ——————.
(*f*) If b_2 becomes smaller, P will become ——————.
(*g*) If c_1 becomes smaller, P will become ——————.
(*h*) If c_2 becomes smaller, P will become ——————.

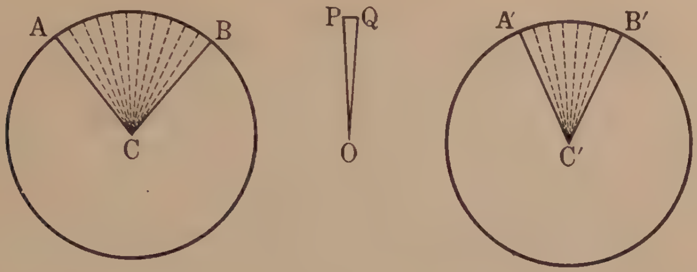
14. Given the formula $N = \frac{ab}{c} - d(e - f) + \frac{1.414}{g + \frac{1}{h}}$.

Answer larger, smaller, or you cannot tell to the following.

(*a*) If a becomes larger, N will become ——————.
(*b*) If b becomes smaller, N will become ——————.
(*c*) If c becomes larger, N will become ——————.
(*d*) If d becomes larger, N will become ——————.
(*e*) If e becomes smaller, N will become ——————.
(*f*) If f becomes larger, N will become ——————.
(*g*) If g becomes smaller, N will become ——————.
(*h*) If h becomes smaller, N will become ——————.
(*i*) If h becomes larger, N will become ——————.

270. Measurement of Angles.—A direct application of ratio and variation to geometry is seen in the measurement of angles. For many purposes it is desirable to deal with magnitudes by taking a certain part of the magnitude, called a unit, and see how many times this unit is contained in the given magnitude. This is especially useful in proving some very important theorems. To measure a given quantity is to find the number of times a given unit is contained in the given quantity. A degree of arc is $\frac{1}{360}$th part of a circle. A quadrant is $\frac{1}{4}$ of a circle and contains 90°.

271. **Theorem.**—In the same circle, or in equal circles, two central angles have the same ratio as their intercepted arcs.



Given the equal circles C and C', central $\angle$s ACB and $A'C'B'$, and arcs AB and $A'B'$.

To prove that $\dfrac{\angle ACB}{\angle A'C'B'} = \dfrac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}$.

Proof.—Let the arc PQ, with radius OP equal to OA, be a common unit of measure of the arcs AB and $A'B'$. Suppose this unit, when applied to the arcs, is contained **m** times in $\overset{\frown}{AB}$ and **n** times in $\overset{\frown}{A'B'}$.

Then $\dfrac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \dfrac{m}{n}$. (Two quantities of the same kind have the same ratio as their numerical measure.)

Draw radii from C and C' to the points of division of the arcs.
Then these unit central angles will all be equal. Why?
There will be **m** equal angles in $\angle ACB$ and **n** in $\angle A'C'B'$.

Hence, $\dfrac{\angle ACB}{\angle A'C'B'} = \dfrac{m}{n}$. Why? $\therefore \dfrac{\angle ACB}{\angle A'C'B'} = \dfrac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}$. *Ax.* 6.
Q.E.D.

In case the arcs AB and $A'B'$ have no common unit of measure, *i.e.*, a unit arc that is contained in AB may not be contained in $A'B'$, they are incommensurable. The proof for this case may be given if desired.

272. **Corollary 1.**—The number of degrees in the central angle is the same as the number of arc degrees in the intercepted arc.

273. **Corollary 2.**—A central right angle intercepts a quadrant of arc.

274. **Segment.**—A segment of a circle is the part of a circle bounded by a chord and its subtended arc.

275. **Inscribed Angle.**—An inscribed angle of a circle is an angle whose vertex is on the circumference and whose sides are chords.

276. **Angles Inscribed in a Segment.**—An angle inscribed in a segment is an angle whose vertex is on the arc of the segment and whose sides are chords drawn to the ends of the chord of the segment.

277. Theorem.—An inscribed angle is measured by one-half the arc intercepted by its sides. There are three cases to be considered.

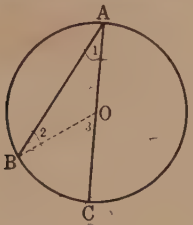


Fig. 1.

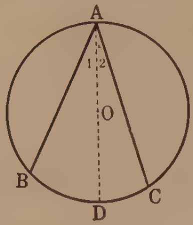


Fig. 2.

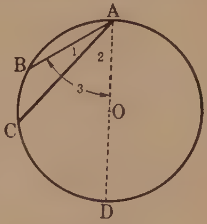


Fig. 3.

Case 1.—When one side of the angle is a diameter, as in Fig. 1.

Given the circle O, the inscribed $\angle BAC$ and AC a diameter.

To prove that $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Proof.—Draw the radius OB. Then $\triangle AOB$ is isosceles. Why?

Hence, $\angle 1 = \angle 2$. Why?
$\angle 3 = \angle 1 + \angle 2$. §129.
$\therefore \angle 3 = 2 \angle 1$ or $\frac{1}{2} \angle 3 = \angle 1$.
But $\angle 3 \underset{=}{m} \overset{\frown}{BC}$. §272.
$\therefore \angle 1 \underset{=}{m} \frac{1}{2} \overset{\frown}{BC}$. Why?
Q.E.D.

Case 2.—When the center of the circle is within the angle, as in Fig. 2.

Hint.—*Draw a diameter from the vertex of the given angle and refer to case 1.*

Case 3.—When the center of the circle is without the angle, as in Fig. 3.

278. **Corollary** 1.—Angles inscribed in the same segment, or in equal segments, are equal.

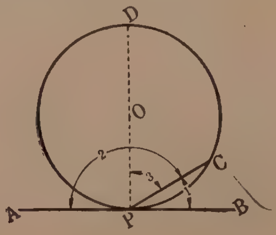
279. **Corollary** 2.—An angle inscribed in a semicircle is a right angle.

280. **Corollary** 3.—Angles inscribed in a segment larger than a semicircle are acute.

281. **Corollary** 4.—Angles inscribed in a segment smaller than a semicircle are obtuse.

282. **Corollary** 5.—The opposite angles of an inscribed quadrilateral are supplementary.

283. **Theorem.**—An angle formed by a tangent and a chord is measured by one half the intercepted arc.



Given AB, a tangent, and PC, a chord of circle O.

To prove that $\angle 1 \overset{m}{=} \frac{1}{2} \overset{\frown}{PC}$,

and $\angle 2 \overset{m}{=} \frac{1}{2} \overset{\frown}{PDC}$.

Proof.—Draw the diameter from P.

Then $\angle DPB$ is a rt. $\angle$. Why?

$\angle 3 \overset{m}{=} \frac{1}{2} \overset{\frown}{DC}$, Why?

and $\angle DPB \overset{m}{=} \frac{1}{2} \overset{\frown}{DCP}$.

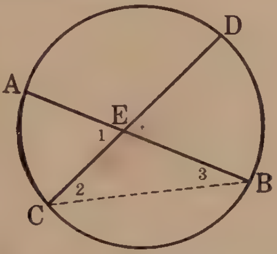
For $\overset{\frown}{DCP}$ is a semicircle and contains 180°.

Subtracting, $\angle 1 \overset{m}{=} \frac{1}{2} \overset{\frown}{PC}$.

But $\angle 2$ is the supplement of $\angle 1$. Why?

$\therefore \angle APC \overset{m}{=} \frac{1}{2} \overset{\frown}{PDC}$. *Q.E.D.*

284. **Theorem.**—The angle formed by two chords intersecting within a circle is measured by one-half the sum of the arcs intercepted between the sides of a pair of vertical angles.



Given the chords AB and CD intersecting at E forming $\angle 1$.

To prove that $\angle 1 \overset{m}{=} \frac{1}{2} (\overset{\frown}{AC} + \overset{\frown}{BD})$.

Proof.—Draw the chord CB.

Then $\angle 1 = \angle 2 + \angle 3$. §129.

But $\angle 2 \overset{m}{=} \frac{1}{2} \overset{\frown}{BD}$, Why?

and $\angle 3 \overset{m}{=} \frac{1}{2} \overset{\frown}{AC}$. Why?

Substituting, $\angle 1 \overset{m}{=} \frac{1}{2} \overset{\frown}{AC} + \frac{1}{2} \overset{\frown}{BD}$.

$\therefore \angle 1 \overset{m}{=} \frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD})$. Why? *Q.E.D.*

285. **Theorem.**—An angle formed by two secants, by a tangent and a secant, or by two tangents meeting without the circle, is measured by one half the difference of the intercepted arcs.

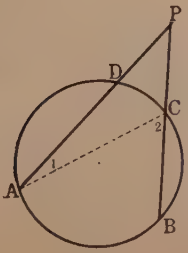


FIG. 1.

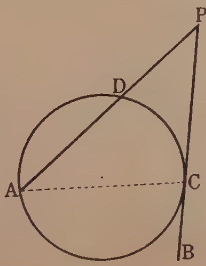


FIG. 2.

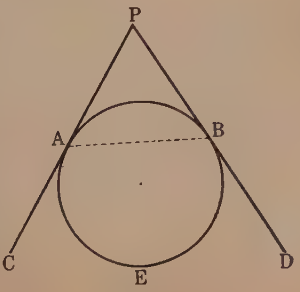


FIG. 3.

1. **Given** $\angle P$ formed by the secants AP and BP, as in Fig. 1.

To prove that $\angle P \overset{m}{=} \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{DC})$.

Proof.—Draw the chord AC.

Then $\angle 2 = \angle 1 + \angle P$. *Why?*

$\therefore \angle P = \angle 2 - \angle 1$. *Ax.* 2.

But $\angle 2 \overset{m}{=} \frac{1}{2}\overset{\frown}{AB}$ and $\angle 1 \overset{m}{=} \frac{1}{2}\overset{\frown}{DC}$. *Why?*

Substituting, $\angle P \overset{m}{=} \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{DC})$. *Q.E.D.*

2. **Given** $\angle P$ formed by the secant AP and tangent BP, as in Fig. 2.

To prove that $\angle P \overset{m}{=} \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{DC})$.

The proof is left to the student.

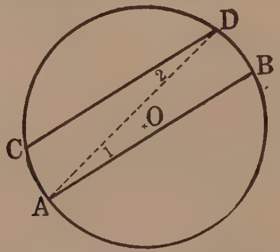
3. **Given** $\angle P$ formed by the tangents PC and PD, as in Fig. 3.

To prove that $\angle P$ is measured by $\frac{1}{2}(\overset{\frown}{AEB} - \overset{\frown}{AB})$.

The proof is left to the student.

286. **Corollary.**—In Fig. 3, the number of degrees in $\overset{\frown}{AB}$ and $\angle P$ together = 180°.

287. **Theorem.**—Two parallel chords of a circle cut off equal arcs.



Given AB and CD two $\parallel$ chords in the circle O.

To prove that $\overset{\frown}{AC} = \overset{\frown}{BD}$.

Proof.—Draw the chord AD.

Then $\angle 1 = \angle 2$. §119

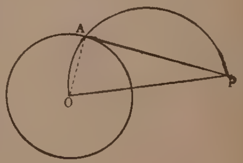
But $\angle 1 \overset{m}{=} \frac{1}{2}\overset{\frown}{BD}$,

and $\angle 2 \overset{m}{=} \frac{1}{2}\overset{\frown}{AC}$. *Why?*

Hence, $\frac{1}{2}\overset{\frown}{AC} = \frac{1}{2}\overset{\frown}{BD}$ or $\overset{\frown}{AC} = \overset{\frown}{BD}$. *Axs.* 7 and 3.

Q.E.D.

288. **Problem.**—To draw a tangent to a given circle from a point outside the circle.



Given the circle O and point P outside the circle.

To construct a tangent from P to the circle O.

Construction.—Draw OP. On OP as a diameter construct the semi-circle OAP. Draw AP. AP is the tangent required.

Proof.—Draw OA.

Then $\angle OAP$ is a rt. $\angle$ §279.

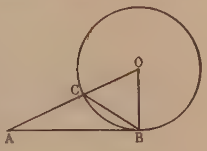
Hence $AP \perp OA$.

$\therefore AP$ is tangent to the circle O. §195. *Q.E.D.*

PROBLEMS

1. A chord of a circle forms equal angles with the tangents at the points of tangency.

2. Given O the center of a circle and AB a tangent. Prove that $\angle ABC = \frac{1}{2} \angle O$.



PROB. 2.

3. If a hexagon $ABCDEF$ is inscribed in a circle the sum of the angles A, C, and E is equal to the sum of the angles B, D, and F.

4. If a circle is circumscribed about an isosceles triangle, the tangents drawn through the vertices form another isosceles triangle. §286.

5. Two sides and the included angle of a quadrilateral are fixed. If the angle opposite the included angle is its supplement, find the locus of the vertex of the angle opposite the given angle. §282.

6. Two secants meet without a circle and intercept arcs which are respectively $\frac{4}{9}$ and $\frac{4}{15}$ of the circumference. Find the number of degrees in the angle formed by the secants.

7. Two secants which meet without a circle form an angle of 70°. If one of the intercepted arcs is 160°, what is the other?

8. The arc intercepted between two tangents drawn from an external point is equal to one-ninth of the circumference; find the angle formed by the tangents.

9. Two tangents drawn to a circle form an angle of 75°. Find the number of degrees in the arc between the two points of contact.

10. If two chords of a circle are perpendicular to each other, the sum of two opposite arcs is equal to a semi-circumference.

11. The interior angle of an inscribed quadrilateral is equal to the exterior angle formed by producing a side through the opposite vertex.

12. Two equal chords drawn from the point of contact make equal angles with the tangent.

13. An angle formed by a tangent and a chord is equal to the angle inscribed in the segment opposite.

14. One of the arcs made by a diameter and a chord drawn perpendicular to it is equal to 42°. How many degrees in each of the other arcs?

15. A chord divides a circle into two segments? An angle inscribed in one segment is 112°. How many degrees in the angle inscribed in the other segment?

16. A chord divides a circle so that an angle inscribed in one segment equals three times an angle inscribed in the other segment. Find the number of degrees in each arc of the segments.

17. Two intersecting chords make an angle of 40°. If one of the arcs contains 36°, how many degrees in the arc intercepted by the vertical angle?

18. Two intersecting chords intercept opposite arcs of 25° and 101°. Find the angle between the chords.

19. Two secants intercept arcs of 20° and 84°. Find the angle made by the secants.

20. The angle formed by two secants intersecting outside the circle is 15°. If one of the intercepted arcs is 120°, find the other arc.

21. The angle formed by two tangents is 62°. Find the number of degrees in the arcs into which they divide the circumference.

22. Find the angle formed by two tangents which intercept an arc of 90°.

23. The angle formed by a tangent and a secant is 35°. If the larger intercepted arc is 110°, find the other arc.

24. What angle is formed by two tangents which intercept an arc of 120°?

25. Two sides of an inscribed triangle subtend arcs of 122° and 156° respectively. Find the number of degrees in each angle of the triangle.

26. The base of an inscribed isosceles triangle subtends an arc of 92°. Find the angles of the triangle.

27. The bases of an inscribed trapezoid subtend arcs of 82° and 124° respectively. Find the angles of the trapezoid and also the angle formed by the nonparallel sides produced.

28. Find the angles between the diagonals of an inscribed trapezoid if one of the legs subtends an arc of 68°.

29. If the angle formed by two tangents is 78°, what is the angle formed by one of the tangents and the chord joining the two points of contact?

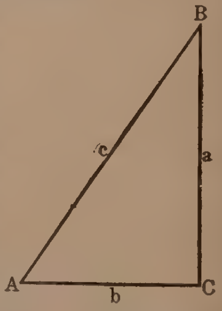
30. The bisectors of the angles A, B, and C of an inscribed triangle meet the circumference at D, E, and F respectively. If the angle A is equal to 84°, and the angle B is equal to 50°, find the number of degrees in the angle EDF.

31. The vertices of an inscribed quadrilateral divide the circumference into arcs which are to each other as 1, 2, 3, and 4. Find all the angles of the quadrilateral. (Let the arcs $= x$, $2x$, $3x$, etc.)

32. The diagonals AC and BD of an inscribed quadrilateral intersect at E. If the sides AB and BC subtend arcs of 43° and 97° respectively, and the angle between the diagonals is 105°, how many degrees are there in each angle of the quadrilateral?

NATURAL FUNCTIONS OF ACUTE ANGLES

289. **Natural Functions of Angles.**—Another direct application of ratio is found in defining the natural functions of angles. For the present we will confine ourselves to the functions of acute angles as applied to the solution of right triangles. Later on we will need the functions of obtuse angles, for the solution of oblique triangles.



Let ABC be a right triangle; a, b, and c the lengths of the sides opposite the angles A, B, and C respectively. There are six simple relations called functions of an angle namely; sine (sin), cosine (cos), tangent (tan), cotangent (cot), secant (sec), and cosecant (csc).

The word, function, as here used is justified by § 267. The cosine, cotangent, and cosecant are called co-functions.

The functions of A and B are defined by the following ratios.

$$\sin A = \frac{a}{c} = \cos B;\ \cot A = \frac{b}{a} = \tan B;$$

$$\cos A = \frac{b}{c} = \sin B;\ \sec A = \frac{c}{b} = \csc B;$$

$$\tan A = \frac{a}{b} = \cot B;\ \csc A = \frac{c}{a} = \sec B.$$

Angles A and B are complementary. Hence the functions of an angle are the same as the co-functions of its complement.

PROBLEMS

The following table is computed from the natural functions of the given angles. It is to be used in solving the following problems.

Degrees	sin	cos	tan
15°	.259	.966	.268
30°	.500	.866	.577
40°	.643	.766	.839
45°	.707	.707	1.000
50°	.766	.643	1.192
60°	.866	.500	1.732
75°	.966	.259	3.732

1. If $\frac{b}{40} = \sin 30°$, find b.

2. $\frac{a}{b} = \tan 60°$, when $b = 40$, find a.

3. If $\frac{a}{40} = \sin 30°$, find a.

4. If $\frac{b}{37} = \tan 30°$, find b.

5. If $\frac{36}{c} = \sin 30°$, find c.

6. If $\frac{40}{b} = \frac{1}{\sin 75°}$, find b.

7. If $\frac{40}{a} = \frac{1}{\cos 15°}$, find a.

8. If $\frac{40}{b} = \frac{1}{\cos 75°}$, find b.

9. If $\frac{40}{a} = \frac{1}{\sin 75°}$, find a.

10. If $\frac{b}{320} = \cos 50°$ find b.

11. If $\frac{b}{320} = \sin 40°$, find b.

12. If $\frac{b}{14} = \tan 45°$, find b.

13. If $\frac{30}{a} = \tan 45°$, find a.

14. If $\frac{b}{30} = \sin 45°$, find b.

15. If $\frac{a}{30} = \tan 75°$, find a.

16. If $\frac{73}{b} = \tan 60°$, find b.

17. If $\frac{108}{b} = \tan 15°$, find b.

18. If $\frac{304}{a} = \frac{1}{\cos 60°}$, find a.

19. If $\frac{1}{\tan A} = 1$, find A.

20. If $\cos A = \frac{22.98}{30}$, find A.

290. **Proportion.**—A proportion is the statement of the equality of two ratios.

Thus $\frac{a}{b} = \frac{c}{d}$, also written $a:b = c:d$, is a proportion.

The **extremes** are the first and last terms, as **a** and **d**.

The **means** are the second and third terms as, **b** and **c**.

The **antecedents** are the first and third terms as, **a** and **c**.

The **consequents** are the second and fourth terms, as **b** and **d**.

A **mean proportional** is the second or third term of a proportion in which the means are the same, as **b** in $a:b = b:c$.

A **third proportional** is the last term of a mean proportion, as **c** in the above.

A **fourth proportional** is the last term of any proportion, as **d** in $a:b = c:d$.

A **continued proportion** is a series of equal ratios in which the consequent of any ratio is the antecedent of the one following, as $a:b = b:c = c:d$, etc.

291. **Theorem.**—In any proportion the product of the extremes is equal to the product of the means.

Given $\frac{a}{b} = \frac{c}{d}$ or $a:b = c:d$.

To prove that $ad = bc$.

Proof. $\frac{a}{b} = \frac{c}{d}$. Given.

Clearing of fractions, $ad = bc$. *Ax.* 3.

Q.E.D.

292. **Corollary 1.**—A mean proportional is equal to the square root of the product of the extremes.

293. **Corollary 2.**—If any three terms of one proportion are equal to the corresponding three terms of another proportion, the fourth terms are equal.

Suggestion.—Apply §291 and *Ax.* 6.

294. **Theorem.**—If the product of two quantities is equal to the product of two other quantities, one pair may be made the extremes and the other pair the means. (Converse of the theorem in §291.)

Given $ad = bc$.

To prove that $a:b = c:d$.

Proof. $ad = bc$. Given.

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$ or $\frac{a}{b} = \frac{c}{d}$. *Ax.* 4.

Q.E.D.

295. **Theorem.**—If four quantities are in proportion, they are in proportion by alternation, that is, the first is to the third as the second is to the fourth.

Given $a:b = c:d$.

To prove that $a:c = b:d$.

Proof. $a:b = c:d$. Given.

Then $ad = bc$. §291.

Dividing by cd, $\frac{a}{c} = \frac{b}{d}$ or $a:c = b:d$.

Q.E.D.

296. **Theorem.**—If four quantities are in proportion, they are in proportion by inversion; that is, the second is to the first as the fourth is to the third.

The proof is left to the student.

297. **Theorem.**—If four quantities are in proportion, they are in proportion by composition; that is, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.

Given $a:b = c:d$.

To prove that $a + b:b = c + d:d$.

Proof. $\frac{a}{b} = \frac{c}{d}.$ Given.

Adding 1 to each member of the equation, $\frac{a}{b} + 1 = \frac{c}{d} + 1.$

Simplifying, $\frac{a+b}{b} = \frac{c+d}{d}$ or $a + b:b = c + d:d.$ *Q.E.D.*

Prove that $a + b:a = c + d:c.$

298. **Theorem.**—If four quantities are in proportion they are in proportion by division; that is, the difference of the first and second is to the second as the difference of the third and fourth is to the fourth.

Suggestion.—The proof is the same as in §297, except that 1 is subtracted from each member of the equation.

Prove that $b - a:b = d - c:d.$

299. **Theorem.**—If four quantities are in proportion, they are in proportion by composition and division; that is, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

Suggestion.—Divide the result of §297 by the result of §298.

PROBLEMS

1. If $45:68 = 90:x$, find x.

2. If $6:3 = x:7$, find x.

3. If $x:1\frac{1}{2} = 1\frac{5}{7}:1\frac{4}{5}$, find x.

4. If $3:x = 7:42$, find x.

5. Given $m:n = p:q$. Write this proportion, (*a*) by alternation, (*b*) by inversion. (*c*) by composition, (*d*) by division, (*e*) by composition and division.

6. Given $2:3 = 6:9$. Write as in Prob. 5.

7. Given $6 \times 4 = 3 \times 8$. Write eight proportions using these numbers.

8. Given $x + y:x - y = 5:3$. Write (*a*) by composition. (*b*) By division. (*c*) By composition and division.

300. **Theorem.**—Like powers and like roots of the terms of a proportion are in proportion.

Let the student supply the proof.

301. **Theorem.**—In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents, as any antecedent is to its consequent.

Given $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} =$ etc.

To prove that $\frac{a + c + e + g + \ . \ . \ .}{b + d + f + h + \ . \ . \ .} = \frac{a}{b} = \frac{c}{d}$, etc.

Proof.—Let $\frac{a}{b} = r, \frac{c}{d} = r, \frac{e}{f} = r,$ and $\frac{g}{h} = r.$

Then $a = br$, $c = dr$, $e = fr$, and $g = hr$. Why?

Adding, $a + c + e + g = br + dr + fr + hr$, or $(b + d + f + h)r.$

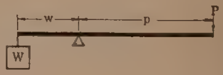
Dividing by $b + d + f + h$, $\frac{a + c + e + g}{b + d + f + h} = r.$

But $\frac{a}{b} = \frac{c}{d} = r$, $\therefore \frac{a+c+e+g}{b+d+f+h} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$ etc. Why?

Q.E.D.

PROBLEMS

1. Find the mean proportional between 8 and 2.
2. Find the third proportional to 3 and 9.
3. Find the fourth proportional to 6, 10, and 12.
4. Find the mean proportional between -20 and -5.
5. Find x if $5 + x:1 - x = 3:5$.
6. Find the third proportional to $\frac{2}{3}$ and $\frac{4}{5}$.
7. Find the fourth proportional to a^3, ab, and $3a^2$.
8. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{ac}{bd} = \frac{c^2}{d^2}$.
9. What number must be added to each of the terms of $\frac{2}{5}$ to make the value of the fraction $\frac{3}{4}$?
10. What number must be added to each of the numbers 3, 7, 15 and 25 so that the resulting numbers will be in proportion?
11. The horsepower generated by a stream falling over a dam is proportional to the height of the dam. If a dam 10 feet high generates 200 horse power, how high must a dam be to generate 750 horse power?
12. A window makes the best appearance when its height is to its width as 7:5. If a window is to be 4′ 6″ high, how wide should it be?
13. If a steeple in Washington, D. C., 100 feet high casts a shadow 40 ft. long, how high is the Washington Monument which casts a shadow 222 feet long at the same time?
14. If a piece of ground can be divided into 50 building lots each 25 feet wide, how many lots each $31\frac{1}{4}$ feet wide would it make?
15. If x is a mean proportional between y and z, prove that $y - 2x:x - 2z = 2y - 3x:2x - 3z$.
16. If 3000 shingles 3 inches wide will cover a roof, how many shingles 4 inches wide would be needed to cover the same roof?
17. The law of levers is given by the proportion $\frac{P}{W} = \frac{w}{p}$.



Solve for W, for w, and also for p.
Find P, if $W = 250$ lbs., $w = 10$ ft., and $p = 25$ ft.

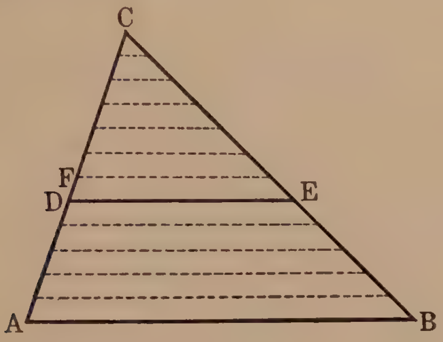
18. Find w, if $P = 400$ lbs., $W = 1000$ lbs., and $p = 5$ ft.
19. A brakeman pulls with a force of 125 pounds on a brake wheel 20 inches in diameter. The force is communicated to the brake chain by means of an axle $2\frac{1}{2}$ inches in diameter. Find the pull on the brake chain.
20. The pressure P exerted by a book press is connected with the force F, r the radius of the wheel, and h the pitch of the screw by the following proportion:

$$\frac{F}{P} = \frac{h}{2\pi r} \cdot \left(\pi = \frac{22}{7}\right).$$

What pressure is exerted by a force of 28 lb. applied to the wheel of such a press 10 inches in diameter if the threads of the screw are $\frac{1}{2}$ inch apart?

302. Lines Divided Proportionally.—Two lines are said to be divided proportionally, if any two segments of one line are in the same ratio as the corresponding segments of the other line.

303. **Theorem.**—If a line is parallel to one side of a triangle, it divides the other two sides proportionally.



Given $\triangle ABC$ and $DE \parallel$ to AB.

To prove that $CD:DA = CE:EB$.

Proof.—Take a unit of measure, as DF, which will divide exactly into CD and DA.

Suppose this unit is contained in CD **m** times and in DA **n** times.

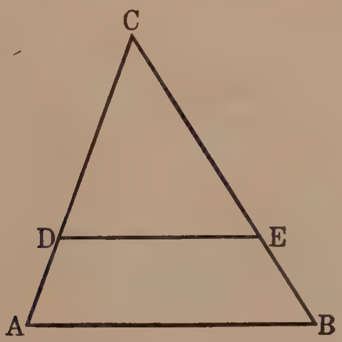
Then $\frac{CD}{DA} = \frac{m}{n}$. $\left(\begin{array}{l}\text{Two quantities of the same kind have the same}\\ \text{ratio as their numerical measure.}\end{array}\right)$.

Now draw lines $\parallel$ to AB through these points of division. These lines will divide CB into **m** + **n** equal parts, CE into **m** parts and EB into **n** parts. §162.

Then $\frac{CE}{EB} = \frac{m}{n}$. Why? $\therefore \frac{CD}{DA} = \frac{CE}{EB}$. Why?

Q.E.D.

304. **Theorem.**—If a line is parallel to one side of a triangle and intersects the other two sides, one of the intersected sides is to a segment of that side as the other side is to its corresponding segment.



Given $\triangle ABC$ and $DE \parallel AB$.

To prove that (1) $AC:DC = BC:EC$.

(2) $AC:AD = BC:BE$.

Proof.—(1) $AD:DC = BE:EC$. §303.

Then $AD + DC : DC = BE + EC : EC$. §297.

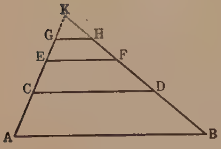
$\therefore AC : DC = BC : EC$. *Axs.* 12 and 5.

Q.E.D.

(2) The proof is similar to (1).

305. **Corollary.**—If two lines are cut by a series of parallel lines, the corresponding segments are proportional.

Suggestion.—Extend AG and BH to intersect at K.



Then, $\dfrac{AC}{BD} = \dfrac{CK}{DK}$, §303 and §295.

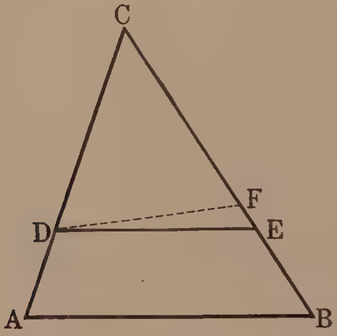
and $\dfrac{CE}{DF} = \dfrac{CK}{DK}$, §304 and §295.

$\therefore \dfrac{AC}{BD} = \dfrac{CE}{DF}$. *Ax.* 6.

In the same manner, $\dfrac{CE}{DF} = \dfrac{EG}{FH}$.

$\therefore \dfrac{AC}{BD} = \dfrac{CE}{DF} = \dfrac{EG}{FH}$. *Q.E.D.*

306. **Theorem.**—(Converse of §303.) If a line divides two sides of a triangle proportionally, it is parallel to the third side.



Given $\triangle ABC$ and DE drawn so that $AC : AD = BC : BE$.

To prove that $DE \parallel AB$.

Proof.—If DE is not $\parallel AB$, draw $DF \parallel AB$.

Then $AC : AD = BC : BF$. §303.

But $AC : AD = BC : BE$. Given.

$\therefore BF = BE$. §293.

Hence F falls on E and DF coincides with DE. Why?

$\therefore DE \parallel AB$. Why?

Q.E.D.

PROBLEMS

1. In the figure of §303, if $AD = 4$, $CD = 6$, $CE = 3$, find BE.
2. In the same figure if $CD = 6$, $CE = 3$, and $BE = 4$, find AD.
3. In the figure of §304, if $AC = 5$, $CE = 12$, and $BE = 8$, find AD and CD.

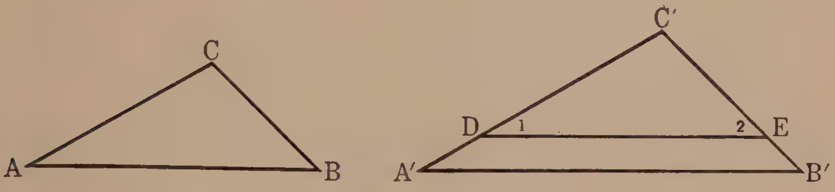
4. In the same figure if $AD:CD = 1:3$, and $BC = 20$, find BE and CE.

5. In the figure of §305, if $AC:CE:EG = 3:2:1$, $BH = 12$, and $FH = 2$, find BD and DF.

307. **Similar Polygons.**—Two polygons are similar if their corresponding angles are equal and their corresponding sides are proportional. The ratio between any two corresponding sides is the ratio of similitude.

Illustration.—Two or more triangles are similar if they are mutually equiangular and their corresponding sides are proportional.

308. **Theorem.**—Two mutually equiangular triangles are similar.



Given $\triangle$s ABC and $A'B'C'$ in which $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$.

To prove $\triangle ABC$ similar to $\triangle A'B'C'$.

Since the corresponding angles of the two triangles are equal, it remains only to prove that $AB:A'B' = BC:B'C' = CA:C'A'$.

Proof.—Place $\triangle ABC$ upon $\triangle A'B'C'$ so that $\angle C$ coincides with its $=$ $\angle C'$, and AB takes the position of DE.

Then $\angle 1 = \angle A'$ and $DE \parallel A'B'$. Why?

Hence $C'D:C'A' = C'E:C'B'$. Why?

Substituting CA for $C'D$ and CB for $C'E$ we get, $CA:C'A' = CB:C'B'$.

In like manner, by placing $\angle B$ on $\angle B'$, we prove that $AB:A'B' = BC:B'C'$.

Hence $AB:A'B' = BC:B'C' = CA:C'A'$. *Ax.* 6.

$\therefore$ The $\triangle$s ABC and $A'B'C'$ are similar, by the definition of similar $\triangle$s.

309. **Corollary** 1.—If a line is parallel to one side of a triangle, the small triangle formed is similar to the original triangle.

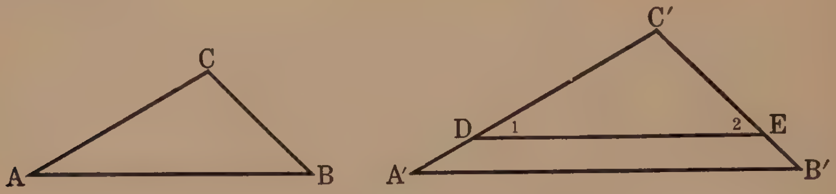
310. **Corollary** 2.—If two triangles have two angles of one equal to the corresponding two angles of the other, they are similar.

311. **Corollary** 3.—If two right triangles have an acute angle of one equal to the corresponding acute angle of the other, they are similar.

312. **Corollary** 4.—If two triangles are each similar to the same triangle they are similar to each other, for they are mutually equiangular.

313. **Corollary** 5.—Two triangles are similar if their corresponding sides are parallel or perpendicular, for they are mutually equiangular. §§126 and 127.

314. **Theorem.**—Two triangles are similar if an angle of one equals an angle of the other and the sides including these equal angles are proportional.



Given the $\triangle$s ABC and $A'B'C'$ in which $\angle C = \angle C'$ and $CA:C'A' = CB:C'B'$.

To prove $\triangle ABC$ similar to $\triangle A'B'C'$.

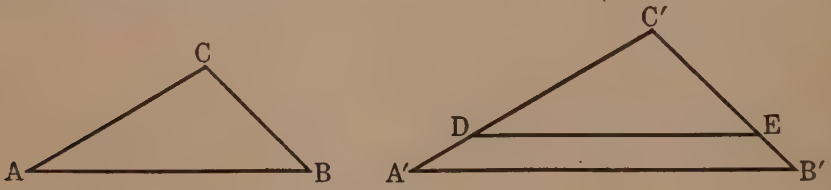
Proof.—Place $\triangle ABC$ upon $\triangle A'B'C'$ so that $\angle C$ coincides with its equal $\angle C'$, and AB takes the position of DE.

Then $C'D:C'A' = C'E:C'B'$. Given.
Hence $DE \parallel A'B'$. §306.
Then $\angle 1 = \angle A'$ and $\angle 2 = \angle B'$. Why?
$\therefore$ $\triangle$s DEC' and $A'B'C'$ are similar.

Substituting $\triangle ABC$ for its equal $\triangle DEC'$, we have $\triangle ABC$ similar to $\triangle A'B'C'$.

Q.E.D.

315. **Theorem.**—Two triangles are similar if their corresponding sides are proportional.



Given $\triangle$s ABC and $A'B'C'$ in which $AB:A'B' = BC:B'C' = CA:C'A'$.

To prove that $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof.—On $C'A'$ take $C'D = CA$ and on $C'B'$ take $C'E = CB$. Draw DE.

Now $C'D:C'A' = C'E:C'B'$. Given.
Hence, $DE \parallel A'B'$. §306.
$\therefore$ $\triangle DEC'$ is similar to $\triangle A'B'C'$. Why?
It remains to prove that $\triangle ABC = \triangle DEC'$.
$AB:A'B' = AC:A'C'$, Given.
and $DE:A'B' = DC':A'C'$. (Since $\triangle$s DEC' and $A'B'C'$ are similar.)
But $DC' = AC$. Hence $AB = DE$. §293.
$\therefore \triangle ABC = \triangle DEC'$. §111.
$\therefore$ $\triangle ABC$ is similar to $\triangle A'B'C'$. *Ax.* 5.

Q.E.D.

PROBLEMS

1. Prove that two isosceles triangles are similar, if the vertex angle of one equals the vertex angle of the other.

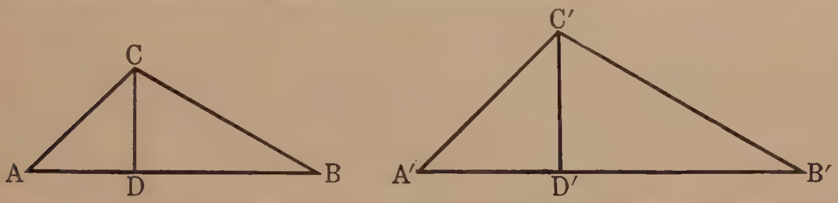
2. Prove that all equilateral triangles are similar; that all squares are similar.

3. Are all rectangles similar?

4. The sides of a triangle are 7, 9, and 12 and the shortest side of a similar triangle is $10\frac{1}{2}$. Find the remaining sides.

5. The sides of a triangle are $3a$, $4a$, and $5a$ and the longest side of a similar triangle is $10a$. Find the remaining two sides. What kind of triangles are these?

316. **Theorem.**—The corresponding altitudes of two similar triangles are to each other as any two corresponding sides.



Given △s ABC and $A'B'C'$, and CD and $C'D'$ two corresponding altitudes.

To prove that $CD:C'D' = AB:A'B' = BC:B'C' = CA:C'A'$.

Proof.—△s ACD and $A'C'D'$ are rt. △s. Why?

$\angle A = \angle A'$.

$\therefore$ △ ACD is similar to △ $A'C'D'$. §311.

$\therefore CD:C'D' = CA:C'A'$. Why?

But $CA:C'A' = AB:A'B' = BC:B'C'$. Why?

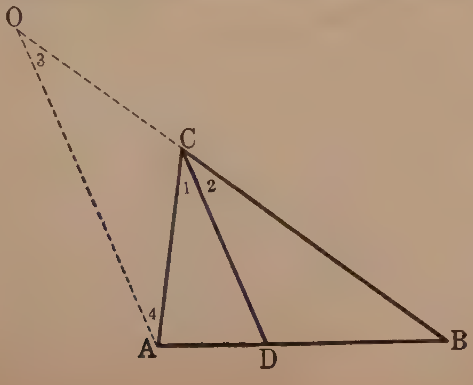
$\therefore CD:C'D' = AB:A'B' = BC:B'C' = CA:C'A'$. *Ax.* 5.

Q.E.D.

317. **Line Segments.** The segments of a line are the parts of the line between the extremities and any point in the line as *A P B.* *AP* and *PB* are the segments of the line *AB*.

But a line is divided externally into segments when the point is on the line extended as *A B P.* *AP* and *PB* are the segments of the line *AB*.

318. **Theorem.**—The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.



Given $\triangle ABC$ and CD the bisector of $\angle C$.

To prove that $AD:DB = AC:CB$.

Proof.—Through A draw $AO \parallel DC$ and intersecting BC, extended, at O.

Then $AD:DB = OC:CB$. §303.

$\angle 1 = \angle 2$. Given.

$\angle 1 = \angle 4$. §119.

$\angle 2 = \angle 3$. §120.

$\angle 3 = \angle 4$. *Ax.* 7.

Hence, $\triangle ACO$ is isosceles and $AC = OC$. Why?

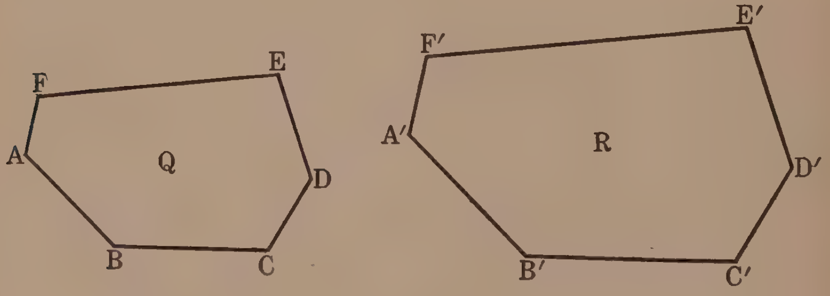
Substituting AC for its equal OC we get, $AD:DB = AC:CB$. *Q.E.D.*

Note.—*The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*

PROBLEMS

1. In figure of §318, if $AD = 4$, $AC = 8$, and $BC = 10$, find DB.
2. If $DB = 2$, $AC = 7$, and $CB = 4$, find AD.
3. If $AB = 5$, $AC = 6$, and $CB = 4$, find AD and DB.
4. If $AB = 24$, $AC = 24$, and $CB = 12$, find AD and DB.

319. **Theorem.**—The perimeters of two similar polygons are to each other as any two corresponding sides.



Given similar polygons Q and R whose perimeters are P and P' respectively.

To prove that $P:P' = AB:A'B' = BC:B'C' =$ etc.

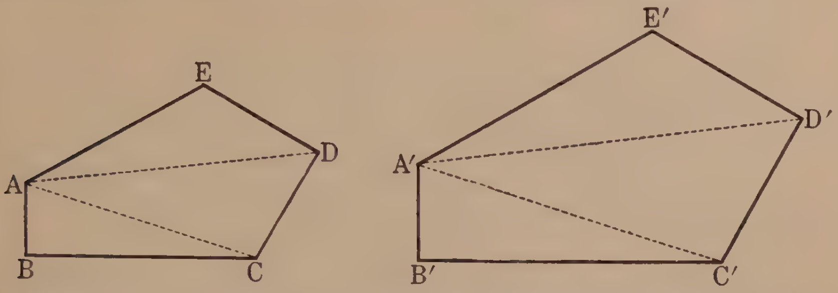
Proof.—$AB:A'B' = BC:B'C' = CD:C'D' = DE:D'E' =$ etc. §307.

$\therefore AB + BC + CD + DE +$ etc.$: A'B' + B'C' + C'D' + D'E' +$ etc. $= AB:A'B' = BC:B'C' =$ etc. §301.

$\therefore P:P' = AB:A'B' = BC:B'C' =$ etc. Why?

Q.E.D.

320. **Theorem.**—Two similar polygons may be divided into the same number of triangles, similar each to each and similar placed.



Given similar polygons AD and $A'D'$ divided into the same number of triangles by diagonals drawn from two corresponding vertices, as A and A'.

To prove that the corresponding △s are similar.

Proof.—$\angle B = \angle B'$ and $AB:A'B' = BC:B'C'$. §307.

Then $\triangle ABC$ is similar to $\triangle A'B'C'$. §314.

Hence $BC:B'C' = AC:A'C'$ and $BC:B'C' = CD:C'D'$. Why?

$\therefore AC:A'C' = CD:C'D'$. *Ax.* 6.

Now $\angle C = \angle C'$, and $\angle BCA = \angle B'C'A'$. Why?

Subtracting, $\angle ACD = \angle A'C'D'$.

$\therefore \triangle ACD$ is similar to $\triangle A'C'D'$. §314.

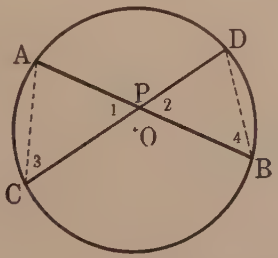
In like manner $\triangle ADE$ can be proved similar to $\triangle A'D'E'$, etc.

∴ The corresponding △s are similar. *Q.E.D.*

321. **Theorem.**—(Converse of §320.) If two polygons are divided into the same number of triangles similar each to each and similarly placed, the polygons are similar.

It must be proved that the corresponding ∠s are equal, and the corresponding sides proportional. Use Fig. of §320.

322. **Theorem.**—If two chords are drawn through a fixed point in a circle, the product of the segments of one chord is equal to the product of the segments of the other chord.



Given circle O and chords AB and CD intersecting at P.

To prove that $AP \times PB = CP \times PD$.

Proof.—Draw AC and BD.

Then $\triangle ACP$ is similar to $\triangle PBD$. §310.

For $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. §§84 and 277.
$\therefore AP:PD = CP:PB.$ Why?
$\therefore AP \times PB = CP \times PD.$ §291.
Q.E.D.

323. **Corollary** 1.—The product of the segments of all chords drawn through the same point in a circle is constant.

PROBLEMS

1. In the figure of §322, if $CP = 5$, $PD = 3$, and $PB = 2$, find AP.

2. In the same figure if $AP = 6$, $PB = 2$, and $PD = 3$, find CP.

3. Two corresponding sides of two similar polygons are 7 and 15. The perimeter of the first polygon is 98 ft. Find the perimeter of the second.

4. The perimeter of two similar polygons are 60 and 90. Find the ratio of any two corresponding sides.

5. The perimeters of two similar polygons are 50 and 125. If the shortest side of the first is 3, find the shortest side of the second.

6. The sides of a polygon are 3, 5, 7, 9, and 10. The shortest side of a similar polygon is 6. Find its perimeter.

7. In the triangle ABC, the altitudes AD and CE intersect at P. Prove that $\triangle APE$ is similar to $\triangle PDC$. Draw a figure.

8. A and B are two points on an arc of a circle. Lines AC, AD, BC, and BD are drawn to the extremities of the arc DC. If AD and BC intersect in P, prove that triangles APC and BPD are similar.

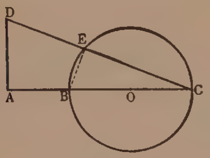
9. Construct two rectangles whose bases are 4 and 8, and altitudes 3 and 6 respectively. Show that they are similar. Draw two corresponding diagonals and prove the rectangles are divided into two triangles similar each to each.

10. If AD and BC of an inscribed quadrilateral are extended to meet at P, prove that the triangles APB and DPC are similar.

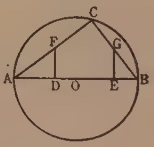
11. Construct a fourth proportional to 2, 3 and 5.

12. Construct two similar triangles in which the ratio of similitude is 3 to 4.

13. Given AC a line passing through the center of circle O. $AD \perp AC$ and DC intersecting the circle at E. Prove that $\triangle ACD$ is similar to $\triangle BCE$.



PROB. 13.



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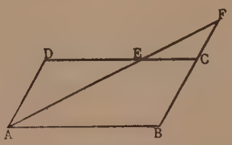


PROB. 15.

14. Given $\triangle ABC$ inscribed in a semicircle. FD and $GE \perp AB$. Prove that $\triangle$s ABC, AFD, and BEG are all similar.

15. $ABCD$ is a parallelogram, E any point in CD and AE extended to meet BC extended to F. Prove that $\triangle$s ADE, EFC and AFB are all similar.

16. The line drawn bisecting any two sides of a triangle forms a triangle similar to the given triangle.

17. If the diagonals of a trapezoid be drawn two similar triangles are formed.

18. In triangle ABC, AD and BE are altitudes. Prove triangles ADC and BEC similar.

19. Two corresponding medians of two similar triangles have the same ratio as any two corresponding sides.

20. A monument casts a shadow 40 feet long when a pole 20 feet high casts a shadow 5 feet long. Find the height of the monument.

21. Find the height of Smith-Memorial in Fairmount Park, if it casts a shadow 213 feet long when a tree 40 feet high casts a shadow 60 feet long.

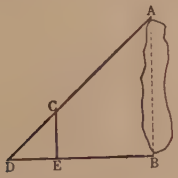
22. If a man five feet tall casts a shadow 98 inches long, what will be the length of the shadow cast by man 6 feet 2 inches tall at the same time?

23. A building casts a shadow 200 feet long, at the same time that a 25 foot telegraph pole casts a shadow 102 feet long. Find the height of the building.

24. Two chords AB and CD are drawn in a circle intersecting at P. Join their extremities by lines and prove the two pairs of triangles formed similar.

25. Prove that two isosceles triangles are similar if a base angle of one equals a base angle of the other.

26. Suppose it is desired to find the width of a lake AB. Select a point D and measure distances DA and DB. Take $DC = \frac{1}{3}\,DA$ and $DE = \frac{1}{3}\,DB$. Prove that $DA:DB = DC:DE$. CE was measured and found to be 350 feet. How long is the lake AB?



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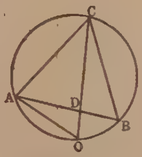
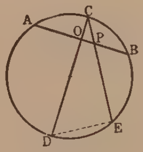


PROB. 27.



PROB. 28.

27. Given arc AC = arc BC. Prove $\triangle$s AOC and ADC similar.

28. AB is a chord in a circle, CD is a diameter $\perp$ to the chord intersecting in O; CE any other chord intersecting AB in P. Prove $\triangle$s COP and CDE similar.

CHAPTER IX

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE

324. **Relation between Simple and Simultaneous Equations.**—(See §325.) If we have one unknown number and one equation we can solve for the unknown as has been seen in Chap. III. But if we have two unknown numbers and only one equation we cannot determine the values of the unknown numbers as will be seen from the equation, $2x - 3y = 4$. For when $x = 1, y = -\frac{2}{3}$, but when $x = 4, y = \frac{4}{3}$. Hence in order to determine the value of the two unknown numbers we must have two equations.

325. **Simultaneous Equations.**—Simultaneous equations are equations in two or more unknown numbers which are satisfied by the same values of the same unknowns. Thus, $x + y = 4$ and $x - y = 2$ are satisfied when $x = 3$ and $y = 1$.

326. **Elimination.**—Elimination is the process by which two equations are combined so as to form one equation containing one unknown. When the equations contain more than two unknowns the process of elimination must be repeated.

327. **Methods of Elimination.**—There are three common methods of elimination: **Addition and Subtraction, Substitution, and Comparison.**

328. **Addition and Subtraction.**—In the method by addition and subtraction, make the coefficients of the same unknown number equal by multiplying or dividing the equations through by some number. Then add the equations if the signs of the equal coefficients are unlike and subtract if they are alike.

Illustration.—Solve for x and y,

$$3x - 2y = 5, \qquad (1)$$

Solution. $$2x + 3y = 12. \qquad (2)$$

Multiplying (1) by 3, $$9x - 6y = 15. \qquad (3)$$

Multiplying (2) by 2, $$4x + 6y = 24. \qquad (4)$$

Adding (3) and (4), $$13x \quad = 39.$$

$$\therefore x = 3.$$

Substituting 3 in (1) for x, $$9 - 2y = 5.$$

Transposing, $$-2y = -4.$$

Therefore $$y = 2.$$

The result is written, $\begin{pmatrix} x, & y \\ 3, & 2 \end{pmatrix}$.

Check both equations by substituting the values of x and y in the original equations as, (1) $9 - 4 = 5$.

(2) $6 + 6 = 12$.

PROBLEMS

Solve by addition and subtraction and check.

1. $5x + y = 11$,
   $3x + 2y = 8$.
2. $x + 3y = -3$,
   $3x - 4y = 17$.
3. $15x - 4y = 3$,
   $9x - 2y = 2$.
4. $\frac{x}{2} + \frac{y}{3} = 10$,
   $\frac{x}{6} - \frac{y}{4} = -2$.
5. $\frac{r}{5} - \frac{s}{6} = \frac{8}{15}$.
   $\frac{r}{12} - \frac{s}{10} = \frac{17}{60}$.

   Result, $\binom{r,\ \ s}{1,\ -2}$.

Hint.—*Clear of fractions first.*

6. $\frac{x}{2} = \frac{y}{3}$,
   $\frac{x}{5} + \frac{2y}{3} = \frac{12}{5}$.
7. $\frac{x+3}{2} - \frac{y-2}{3} = 2$,
   $\frac{3x+1}{4} - \frac{y+1}{3} = 1$.
8. $x + 3y = 11a$,
   $5x - 2y = 4a$.

   Result $\binom{x,\ \ y,}{2a,\ 3a}$
9. $ax + by = m$,
   $ax - by = n$.
10. $x + ay = -1$,
    $y = b(x + 1)$.
11. $x + ay = k$,
    $ax + y = p$.

329. **Substitution.**—In the method by substitution, solve one of the equations for the value of one unknown in terms of the other. Then substitute this expression for the unknown in the other equation and solve the simple equation thus formed.

Illustration.—Solve for x and y,

$$3x - 2y = 5, \qquad (1)$$
$$2x + 3y = 12. \qquad (2)$$

Solution.—Solving for x in (1), $3x = 2y + 5$.

$$\therefore x = \frac{2y+5}{3}. \qquad (3)$$

Substituting $\frac{2y+5}{3}$ for x in (2), $2\left(\frac{2y+5}{3}\right) + 3y = 12$.

Clearing of fractions, $4y + 10 + 9y = 36$.

Transposing and collecting, $13y = 26$.

$$\therefore y = 2.$$

Substituting 2 for y in (3), $x = \frac{4+5}{3} = 3$.

Result $\binom{x,\ y}{3,\ 2}$

Check the result as in §328.

PROBLEMS

Solve by substitution and check.

1. $4x - 5y = 4$,
   $3x - 2y = 10$.
2. $7c + 6d = 20$,
   $2c + 5d = 32$.
3. $3a + 7c = 50$,
   $5a - 2c = 15$.
4. $\frac{x}{3} + \frac{y}{2} = \frac{4}{3}$,
   $\frac{x}{2} + \frac{y}{3} = \frac{7}{6}$.
5. $\frac{x+7}{5} - \frac{2x-y}{4} = 3y - 5$,
   $\frac{5y-7}{2} + \frac{4x-3}{6} = 18 - 5x$.

√ 6. $\dfrac{x+y}{2} + \dfrac{y-x}{2} = 9,$
$\dfrac{x}{2} + \dfrac{x+y}{9} = 5.$

7. $\dfrac{1}{2}x + \dfrac{1}{3}y = \dfrac{7}{12},$
$\dfrac{3}{4}x - \dfrac{2}{3}y = -\dfrac{7}{24}.$

√ 8. $\dfrac{x+3y}{x-y} = 8,$
$\dfrac{7x-13}{3y-5} = 4.$

9. $bx + cy = abc,$
$x = ay.$

10. $2x - 3y = a + b,$
$3x - 2y = a - b.$

330. Comparison.—In the method by comparison, find the values of one of the unknowns in each equation in terms of the other unknown and equate these values.

Illustration.—Solve for x and y,

$$3x - 2y = 5. \tag{1}$$
$$2x + 3y = 12. \tag{2}$$

Solution.—Solve for x in both (1) and (2).

From (1), $x = \dfrac{2y+5}{3}.$ (3) From (2) $x = \dfrac{12-3y}{2}.$ (4)

Equating the values of x in (3) and (4), $\dfrac{2y+5}{3} = \dfrac{12-3y}{2}.$ (5)

Solving equation (5), $4y + 10 = 36 - 9y.$
$\therefore y = 2.$

Substituting 2 for y in (3) or (4), $x = 3.$

Result, $\begin{pmatrix} x, & y \\ 3, & 2 \end{pmatrix}.$

Check as before.

PROBLEMS

Solve by comparison and check.

1. $y - x - 1 = 3,$
$x - 6 = -y.$

2. $5x + 3y = -1,$
$7y + 3x = 15.$

3. $4x - 6y - 42 = 0,$
$5x - 7y - 50 = 0.$

4. $x + 2y = 3\frac{1}{2},$
$x = \frac{1}{3}y.$

5. $\dfrac{x+y}{y-x} = -\dfrac{13}{4},$
$3x - \dfrac{3y+4x}{7} = 2.$

6. $\dfrac{3x-5y}{2} + \dfrac{59}{10} = \dfrac{2x+y}{5},$
$1 - \dfrac{x-2y}{4} = \dfrac{x}{2} + \dfrac{y}{3},$

7. $ax + by = 1,$
$ax - by = 3.$

8. $\dfrac{x+a}{y+a} = \dfrac{x-2a}{y-2a},$
$\dfrac{x-3a}{y-a} = \dfrac{x+3a}{y+a}.$

9. $x - \dfrac{2y-x}{23-x} = x - 9\frac{1}{2},$
$y + \dfrac{y-3}{x-18} = y - 5\frac{2}{3}.$

331. Equations Having the Unknowns in the Denominator.—Simultaneous fractional equations with the unknown numbers in the denominator should be solved without first clearing of fractions.

Illustration.—Solve for x and y,

$$\frac{2}{x} - \frac{3}{y} = -6, \tag{1}$$

Solution. $$\frac{3}{x} + \frac{2}{y} = 17. \tag{2}$$

Multiplying (1) by 2, $$\frac{4}{x} - \frac{6}{y} = -12. \tag{3}$$

Multiplying (2) by 3, $\frac{9}{x} + \frac{6}{y} = 51.$ (4)

Adding, $\frac{13}{x} = 39.$

Clearing of fractions, $13 = 39x.$

$$\therefore x = \frac{1}{3}.$$

Multiplying (1) by 3, $\frac{6}{x} - \frac{9}{y} = -18.$

Multiplying (2) by 2, $\frac{6}{x} + \frac{4}{y} = 34.$

Subtracting, $-\frac{13}{y} = -52.$

Clearing of fractions $-13 = -52y.$ Result, $\begin{pmatrix} x, & y \\ \frac{1}{3}, & \frac{1}{4} \end{pmatrix}$

$$\therefore y = \frac{1}{4}.$$

Note.—*It is always advisable to solve equations of this type by addition and subtraction.*

PROBLEMS

Solve and check the following.

1. $\frac{5}{x} + \frac{6}{y} = 11,$ $\quad \frac{15}{x} - \frac{3}{y} = 12.$

2. $\frac{3}{x} + \frac{8}{y} = 18,$ $\quad \frac{8}{x} - \frac{3}{y} = 11\frac{1}{2}.$

3. $\frac{2}{3x} - \frac{3}{2y} = -1,$ $\quad \frac{5}{3x} + \frac{7}{2y} = 12.$

4. $\frac{4}{5x} - \frac{5}{6y} = -1,$ $\quad \frac{3}{5x} - \frac{1}{6y} = 2.$

5. $\frac{1}{2x} + \frac{2}{3y} = 3,$ $\quad \frac{3}{4x} + \frac{4}{5y} = 3.9.$

6. $\frac{2}{x+1} - \frac{3}{y-1} = -\frac{5}{6},$ $\quad \frac{3}{x+1} + \frac{2}{y-1} = 2.$

7. $\frac{5}{2x-3} + \frac{2}{3y-2} = \frac{41}{8},$ $\quad \frac{6}{2x-3} - \frac{4}{3y-2} = \frac{23}{4}.$

8. $\frac{1}{x} + \frac{1}{y} = a.$ $\quad \frac{1}{x} - \frac{1}{y} = b.$

9. $\frac{a}{x} - \frac{b}{y} = \frac{a}{b}.$ $\quad \frac{b}{x} - \frac{a}{y} = \frac{b}{a}.$

10. $\frac{a}{bx} + \frac{b}{ay} = a + b.$ $\quad \frac{b}{x} + \frac{a}{y} = a^2 + b^2.$

Solve the following by any of the above methods and check.

11. $2x + 7y = 17,$ $\quad 5x - 9y = -37.$

12. $4x - 7y = -1,$ $\quad 3x + 2y = 21.$

13. $3p + 2q = 2,$ $\quad 5q - 4p = 18.$

14. $\frac{2}{3}p - \frac{3}{2}q = -5,$ $\quad \frac{3}{2}p - \frac{2}{3}q = 5.$

15. $3m - 2n = 11,$ $\quad 4m + 5n = 7.$

16. $\frac{x}{5} - \frac{y}{3} = -\frac{19}{90},$ $\quad \frac{x}{3} + \frac{y}{5} = \frac{1}{10}.$

17. $5r + 2s = \frac{8}{3},$ $\quad 7r - 3s = \frac{17}{3}.$

18. $t + 6 = 2u,$ $\quad 3t + 5u - 29 = 2t.$

19. $\frac{5x}{14} - 2y = \frac{23}{2},$ $\quad \frac{3y}{7} - 2x = -11.$

20. $$\frac{x+y}{3}+\frac{5}{6}-\frac{2y-x}{2}=\frac{1}{2},\qquad \frac{x-y}{2}+\frac{7}{6}-\frac{2x-y}{3}=\frac{1}{3}.$$

21. $$\frac{2x+3y}{1-4y}=\frac{4}{9},\qquad \frac{4x-y}{x+y}=\frac{22}{3}.$$

22. $$\frac{x+5}{y-2}=\frac{3}{2},\qquad \frac{x+4}{y-3}=2.$$

23. $$y+1-\frac{2x-3y}{3}=\frac{13}{3},\qquad x+1-\frac{3y-2x}{2}=\frac{9}{2}.$$

24. $$\frac{7m}{2}-\frac{3n}{5}=-\frac{97}{20},\qquad \frac{5m}{3}+\frac{2n}{5}=-\frac{83}{30}.$$

25. $$\frac{1}{x}+\frac{2}{y}=8,\qquad \frac{3}{x}-\frac{4}{y}=-6.$$

26. $$\frac{2}{3x}-\frac{5}{2y}=-3,\qquad \frac{3}{2y}-\frac{5}{3x}=-2.$$

27. $$\frac{9}{x}-\frac{2}{y}=\frac{31}{3},\qquad \frac{2}{x}+\frac{3}{y}=0.$$

28. $$\frac{2}{x-1}+\frac{3}{y+1}=2,\qquad \frac{3}{x-1}-\frac{2}{y+1}=\frac{5}{6}.$$

29. $$\frac{5}{2x+3}-\frac{2}{3y-2}=-\frac{9}{7},\qquad \frac{7}{2x+3}-\frac{3}{3y-2}=-2.$$

30. $$3x-3y=10a,\qquad 3x-2y=10a.$$

31. $$\frac{x+1}{a}+\frac{y+1}{b}=1,\qquad \frac{x+1}{b}+\frac{y+1}{a}=1.$$

332. **Equations Containing Three Unknowns.**—To solve simultaneous equations containing three unknowns, proceed as follows:

1. Select any pair of equations and eliminate one unknown.
2. Select another pair of equations and eliminate the same unknown.
3. Solve these resulting equations for the two unknowns.
4. Substitute in any one of the given equations the two values thus obtained and solve for the third unknown.
5. Test results.

Note.—*There must be as many equations as there are unknown numbers involved.*

Illustration.—Solve for x, y, and z,

$$2x-3y+4z=\ \ 11, \tag{1}$$
$$3x+2y-5z=-\ 8, \tag{2}$$
$$5x-\ \ y+2z=15. \tag{3}$$

First, eliminate z between equations (1) and (3).

Multiplying (3) by 2, $10x-2y+4z=30.$

Using (1), $2x-3y+4z=11.$

Subtracting, $$8x+\ y\qquad =19. \tag{4}$$

Second, eliminate z between equations (2) and (3).

Multiplying (2) by 2, $6x+4y-10z=-16.$

Multiplying (3) by 5, $25x-5y+10z=\ \ 75.$

Adding, $$31x-\ y\qquad =\ \ 59. \tag{5}$$

We now have two equations (4) and (5) involving two unknowns, x and y,

$$8x + y = 19, \qquad (4)$$
$$31x - y = 59. \qquad (5)$$

Adding, $39x = 78.$

$\therefore x = 2.$

Substituting $x = 2$ in (4), $16 + y = 19.$

$\therefore y = 3.$

Substituting $x = 2$ and $y = 3$ in (1), $4 - 9 + 4z = 11.$

$\therefore z = 4.$

Result, $\begin{pmatrix} x, & y, & z \\ 2, & 3, & 4 \end{pmatrix}$

PROBLEMS

Solve the following.

1. $x + y + z = 6,$
$2x - 3y + 2z = 2,$
$5x + 2y - 4z = -3.$

2. $3x + 2y - 5z - 17 = 0,$
$4x - 3y + 4z + 5 = 0,$
$2x + 4y - 3z - 19 = 0.$
Result, $\begin{pmatrix} x, & y, & z \\ 2, & 3, & -1 \end{pmatrix}.$

3. $x + y = \frac{5}{6},$
$y + z = \frac{7}{12},$
$x + z = \frac{3}{4}.$

4. $3x - y + 2z = -13,$
$3y + 2x - z = 2,$
$3z + x + 2y = -9.$
Result, $\begin{pmatrix} x, & y, & z \\ -2, & 1, & -3 \end{pmatrix}.$

5. $3x - 4y = a,$
$3z - 2x = -3a,$
$5y + 4z = 14a.$

6. $\frac{2x}{3} + \frac{3y}{4} - \frac{5z}{4} = -\frac{1}{3},$
$\frac{5x}{6} - \frac{5y}{8} + \frac{3z}{4} = -\frac{19}{40},$
$\frac{3x}{2} + \frac{7y}{5} + \frac{3z}{2} = \frac{1}{60}.$
Result, $\begin{pmatrix} x, & y, & z \\ -\frac{1}{2}, & \frac{1}{3}, & \frac{1}{5} \end{pmatrix}.$

333. Problems Involving Two or Three Unknowns.

Illustrations.—1. The sum of two numbers is 12, and 3 times the first equals the second. Find the numbers.

Solution.—Let x = the first number.
Let y = the second number.
Then $x + y = 12,$
and $3x = y.$
Solving, $x = 3$ and $y = 9.$

Check.—In checking a reading problem, it is always best to check it by using the conditions of the problem rather than the equations formed. If the equations are incorrectly formed but correctly solved the answers would check in the equations, but not by the conditions of the problem. Thus, the sum of 3 and 9 is 12, and 9 is 3 times 3. Hence both conditions of the problem are satisfied.

2. If the numerator of a fraction is increased by 3, the value of the fraction is 1. If the denominator is increased by 1, the value of the fraction is $\frac{1}{2}$. Find the fraction.

Solution.—Let x = the numerator,
and y = the denominator,
Then $\frac{x}{y}$ = the fraction.

By the conditions of the problem, $\frac{x+3}{y} = 1$, (1) and $\frac{x}{y+1} = \frac{1}{2}$. (2)

From (1), $x - y = -3$. (3)

From (2), $2x - y = 1$. (4)

Subtracting, $-x = -4$.

$\therefore x = 4$.

Substituting 4 for x in (3), $4 - y = -3$.

$\therefore y = 7$.

$\therefore$ the fraction $= \frac{4}{7}$.

Check.—By the first condition, if 3 is added to the numerator the fraction becomes $\frac{7}{7}$ or 1. By the second condition, if 1 is added to the denominator, the fraction becomes $\frac{4}{8}$ or $\frac{1}{2}$. Both conditions are satisfied. Therefore the result is correct.

3. A certain number is expressed by two digits. The sum of the digits is 12. If the number is divided by the sum of the digits the quotient is 4. Find the number.

Solution.—Let x = the tens' digit,
and y = the units' digit.
Then $10x + y$ = the number. Why?

By the conditions of the problem $x + y = 12$, (1)

and $\frac{10x + y}{x + y} = 4$. (2)

Clearing (2) of fractions, $10x + y = 4x + 4y$.

Simplifying, $2x - y = 0$.

Equation (1), $x + y = 12$.

Solving simultaneously, $x = 4$ and $y = 8$.

$\therefore$ The number is 48.

Check.—First, the sum of 4 and 8 is 12. Second, 48 divided by the sum of 4 and 8 is 4. Both conditions are satisfied.

4. Find two numbers whose sum is 8 and whose difference is 2.

5. Twice the difference of two numbers is 6 and $\frac{1}{3}$ of their sum is $5\frac{2}{3}$. Find the numbers.

6. The sum of two numbers is 1.3 and 3 times the first less 4 times the second is equal to .4. Find the two numbers.

7. Find two numbers such that if the first is increased by 16 the result will be twice the second, and if the second is diminished by 10 the result will be $\frac{1}{4}$ the first.

8. A certain fraction will become equal to $\frac{1}{2}$ if both numerator and denominator are increased by 1 and will become equal to $\frac{1}{3}$ if the denominator is increased by 2. Find the fraction.

9. If the numerator of a certain fraction be increased by 3 its value will be $\frac{2}{3}$. If the numerator be decreased by 2 and denominator increased by 3 its value will be $\frac{1}{5}$. Find the fraction.

10. There are two fractions whose denominators are 8 and 5 respectively and whose difference is $\frac{1}{40}$. If their denominators are interchanged their sum will be equal to $\frac{11}{8}$. Find the fractions.

11. A certain fraction will equal $\frac{3}{5}$ if 4 is added to the numerator, and $\frac{1}{3}$ if 8 is added to the denominator. Find the fraction.

12. A number consisting of two digits is equal to eight times the sum of its digits. If 45 is subtracted from the number, the resulting number will be composed of the same two digits in reverse order. Find the number.

13. The sum of the two digits of a certain number is equal to 9. The number is equal to 5 times the sum of the digits. Find the number.

14. A number is expressed by three digits. The sum of the first and second digits is 5. The sum of the first and third digits is 6 and the number is equal to 26 times the sum of its digits. Find the number.

15. The tens' digit of a two digit number is equal to $\frac{2}{3}$ of the units' digit. The number increased by 27 will give a number composed of the same two digits in reverse order. Find the number.

16. A number consisting of three digits is equal to 33 times the sum of its digits. The sum of the hundreds' and units' digits is equal to the tens' digit and the tens' digit is 5 more than the units' digit. What is the number?

17. The sum of the angles made by one straight line meeting another straight line is equal to 180°. Find the angles, if one angle is twice the other.

18. The ratio of the angles made by one line meeting another is 2:3. Find the two angles.

19. The acute angles of a right triangle are complementary. Find the angles, if their difference is 25°.

20. The difference between the two acute angles of a right triangle is $a°$. Find the angles.

21. The sum of the angles of a triangle is equal to 180°. If one angle of a triangle is 38° and the second angle decreased by 34° is equal to three times the third angle, find the remaining two angles of the triangle.

22. Find the angles of a triangle, if the difference between the first and second is 10°, and the sum of the second and third is 110°.

23. The first angle of a triangle is equal to $\frac{5}{3}$ the second and 5 times the third. Find the angles.

24. Find the number of degrees in an angle which equals $\frac{2}{3}$ of its complement.

25. Find the number of degrees in an angle which equals $\frac{3}{5}$ of its supplement plus 25°.

26. The sum of two angles of a triangle is equal to 150°. Find these two angles if $\frac{2}{3}$ of one diminished by $\frac{3}{5}$ of the other is equal to 18°.

27. I bought three boxes of berries and 5 boxes of cherries for $1.11 and 5 boxes of berries and 3 boxes of cherries for $1.05. How much per box did each kind cost?

28. A man purchased 10 typewriters for \$680. Some of them were bought for \$60 each and the remainder for \$80 each. How many of each kind did he buy?

29. Five transformers and six distributors can be purchased for \$140. At the same rate six transformers and five distributors will cost \$135. Find the price of each.

30. If $\frac{2}{3}$ of A's money is added to $\frac{1}{5}$ of B's money, the sum is \$72; but if $\frac{1}{6}$ of B's money is added to $\frac{3}{2}$ of A's money the sum is \$145. How much money has each?

Note.—*Problems* 31 *to* 34 *inclusive are formulas given in Mechanics of Materials.*

31. Given the following formulas;

(1) $S = \frac{P}{a}$, (2) $E_1 = \frac{e}{L}$, and (3) $E = \frac{S}{E_1}$.

To find **e** in terms of P, L, a and E.

Dividing (1) by (2), $\frac{S}{E_1} = \frac{\frac{P}{a}}{\frac{e}{L}} = \frac{P}{a} \times \frac{L}{e} = \frac{PL}{ae}$.

But by (3), $\frac{S}{E_1} = E$ $\qquad \therefore E = \frac{PL}{ae}$. (4)

Clearing of fractions, $E.ae = PL$ and $e = \frac{PL}{aE}$.

Another method for obtaining the same result is as follows;

From (3), $E_1 = \frac{S}{E}$ and from (2), $e = E_1L$.

Then $e = \frac{S}{E}L$.

Substituting $\frac{P}{a}$ for S, $e = \frac{\frac{P}{a} \cdot L}{E} = \frac{PL}{aE}$

32. Given (1) $S_1 = \frac{P_1}{a_1}$, (2) $P_1 = P \cos \theta$, (3) $a_1 = \frac{a}{\sin \theta}$, and (4) $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$.

Derive by substitution, the formula, $S_1 = \frac{P}{2a} \sin 2\theta$.

33. Given (1) $\frac{1}{2}(S_1 + S_2)a = P$ and (2) $P = \frac{1}{6}d \cdot \frac{S_1 - S_2}{S_1 + S_2}$.

Derive the formula, $S_2 = \frac{P}{a}\left(1 - \frac{6P}{d}\right)$.

Hint.—*Multiply* (1) *by* $\frac{2}{a}$, *and* (2) *by* $\frac{S_1 + S_2}{P}$. *Then equate the values of* $S_1 + S_2$.

34. The following illustrates the old rule for determining the pitch in butt joints.

Given (1) $S = \frac{P}{t(p - d)}$ and (2) $S = \frac{2P}{\pi d^2}$.

Show that $p = d + \frac{\pi d^2}{2t}$.

DETERMINANTS

334. Definition.—The symbol $\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}$ is called a determinant. It is equal to $2 \times 4 - 5 \times 1$, or 3.

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of the second order and is equal to $a_1b_2 - a_2b_1$.

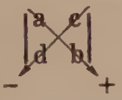
The numbers a_1, a_2, b_1, and b_2 are called elements of the determinant; a_1 and a_2 form a column; a_1 and b_1 form a row; a_1 and b_2 form one diagonal; b_1 and a_2 form the other diagonal.

A determinant of the second order has two rows and two columns, one of the third order has three rows and three columns.

335. Advantage of Determinants.—In addition to the methods already given for solving simultaneous equations, the method by determinants is often more convenient and shorter because the form of the result can be written by inspection. This form must be reduced by evaluating the determinants thus involved.

336. Evaluation of Determinants.—To evaluate a determinant of the second order, take the product of the elements of the diagonal down to the right and subtract from it the product of the elements of the diagonal down to the left. When an element is negative the minus is placed above it to avoid confusion.

Thus,



$= ab - cd.$

If the determinant is of the third order follow the method given in §338. Determinants of higher order than the third will not be treated in this text.

PROBLEMS

Evaluate the following.

1. $\begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix}$. 2. $\begin{vmatrix} 6 & \bar{3} \\ 2 & 1 \end{vmatrix}$. 3. $\begin{vmatrix} 3a & 2b \\ b & 2a \end{vmatrix}$. 4. $\begin{vmatrix} 5m & n \\ 2n & 3m \end{vmatrix}$.

5. $\begin{vmatrix} 2x - 3y & 2x - y \\ x + y & 2x + 2y \end{vmatrix}$. 6. $\begin{vmatrix} 3a & 4\bar{d} \\ 2c & 5b \end{vmatrix}$. 7. $\begin{vmatrix} 3 & 5 \\ 7 & \bar{4} \end{vmatrix}$.

337. Solution of Equations by Determinants.—Given the two simultaneous equations,

$$a_1x + b_1y = k_1, \qquad (1)$$
$$a_2x + b_2y = k_2. \qquad (2)$$

Multiplying (1) by b_2, $a_1b_2x + b_1b_2y = b_2k_1$.

Multiplying (2) by b_1, $\underline{a_2b_1x + b_1b_2y = b_1k_2.}$

Subtracting, $a_1b_2x - a_2b_1x = b_2k_1 - b_1k_2$.

Factoring, $(a_1b_2 - a_2b_1)x = b_2k_1 - b_1k_2$.

$$\therefore \qquad x = \frac{b_2k_1 - b_1k_2}{a_1b_2 - a_2b_1}$$

In the same manner, $y = \dfrac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1}$

These values for x and y may be derived by arranging their coefficients and the constants k_1, and k_2 in rows and columns called determinants as follows:

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

To solve two simultaneous equations by determinants, proceed as follows:

(*a*) Arrange the equations thus, $a_1x + b_1y = k_1.$
$a_2x + b_2y = k_2.$

(*b*) The values of **x** and **y** are fractions whose denominators are the same determinant, formed by the coefficients of **x** and **y** placed in the same order in which they occur in (*a*). The numerator of **x** is the same as the denominator except that the coefficients of **x** are replaced by the constant terms (k_1 and k_2). Likewise the numerator of **y** is the same as the denominator except that the coefficients of **y** are replaced by the constant terms (k_1 and k_2). These simple rules will be clearly seen and understood by comparing them with the above solution of **x** and **y**.

Illustration.—Solve for x and y.

$$2x + 3y = 9,$$
$$3x - 2y = 7.$$

Solution.

$$x = \frac{\begin{vmatrix} 9 & 3 \\ 7 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{-18 - 21}{-4 - 9} = \frac{-39}{-13} = 3.$$

$$y = \frac{\begin{vmatrix} 2 & 9 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{14 - 27}{-4 - 9} = \frac{-13}{-13} = 1.$$

Check the result in the usual way.

Solve the following by determinants. Check.

1. $2x - 3y = 2,\quad 3x + 2y = 16.$
2. $3x - 2y = 8,\quad 5x - 4y = 12.$
3. $4x + 5y = 10,\quad 5y - 4x = 4.$
4. $2x + 5y = 15,\quad 5x - 2y = 23.$
5. $x + 3y = -3,\quad 3x - 4y = 17.$
6. $15x - 4y = 3,\quad 9x - 2y = 2.$
7. $\frac{x}{2} + \frac{y}{3} = 10,\quad \frac{x}{6} - \frac{y}{4} = -2.$
8. $x + 2y = 3\frac{1}{2},\quad x = \frac{1}{3}y.$
9. $4x - 6y - 42 = 0,\quad 5x - 7y - 50 = 0.$

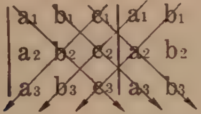
10. $\dfrac{x+y}{y-x} = 5,$

$3x - \dfrac{3y+44}{7} = -\dfrac{11}{7}.$

338. Determinants of the Third Order.—$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a determinant of the third order.

To find the value of a determinant of the third order, proceed as follows: (*a*) Write the determinant repeating the first two columns. (*b*) Find the sum of the products of the elements of the three diagonals down to the right. (*c*) Subtract the sum of the products of the elements of the three diagonals down to the left.

Thus,



$= a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3.$

This method of evaluating a determinant is true only for determinants of the second and third orders.

Illustration.—Find the value of the determinant:

$$\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Repeating the first two columns,

$$\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 3 \\ 3 & 1 & 2 \end{vmatrix} \begin{matrix} 2 & 3 \\ 4 & 5 \\ 3 & 1 \end{matrix} = 20 + 27 + 4 - 15 - 6 - 24 = 6.$$

PROBLEMS

Evaluate the following determinants.

1. $\begin{vmatrix} 1 & 3 & 5 \\ 6 & 2 & 4 \\ 3 & 7 & 1 \end{vmatrix}.$ 2. $\begin{vmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{vmatrix}.$ 3. $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 4 & 2 & 1 \end{vmatrix}.$ 4. $\begin{vmatrix} a & g & h \\ g & h & a \\ h & a & g \end{vmatrix}.$

5. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}.$ 6. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}.$

7. $\begin{vmatrix} x & y & x+y \\ x+y & x & x \\ y & x-y & y \end{vmatrix}.$

339. Solution of Simultaneous Equations of Three Unknown Numbers.

The method is the same as for two unknowns shown by the following illustration.

Given the equations, $a_1x + b_1y + c_1z = k_1,$

$a_2x + b_2y + c_2z = k_2,$

$a_3x + b_3y + c_3z = k_3.$

Solution.

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Illustration.—Solve for x, y, and z.

$$\begin{aligned} 4x - 5y + 2z &= 0, \\ 2x + 3y - z &= 13, \\ 7x - 4y + 3z &= 10. \end{aligned}$$

$$x = \frac{\left|\begin{array}{rrr|rr} 0 & \bar{5} & 2 & 0 & \bar{5} \\ 13 & 3 & \bar{1} & 13 & 3 \\ 10 & \bar{4} & 3 & 10 & \bar{4} \end{array}\right.}{\left|\begin{array}{rrr|rr} 4 & \bar{5} & 2 & 4 & \bar{5} \\ 2 & 3 & \bar{1} & 2 & 3 \\ 7 & \bar{4} & 3 & 7 & \bar{4} \end{array}\right.} = \frac{0 + 50 - 104 - 60 - 0 + 195}{36 + 35 - 16 - 42 - 16 + 30} = \frac{81}{27} = 3.$$

$$y = \frac{\left|\begin{array}{rrr|rr} 4 & 0 & 2 & 4 & 0 \\ 2 & 13 & \bar{1} & 2 & 13 \\ 7 & 10 & 3 & 7 & 10 \end{array}\right.}{27} = \frac{156 + 0 + 40 - 182 + 40 - 0}{27} = \frac{54}{27} = 2.$$

$$z = \frac{\left|\begin{array}{rrr|rr} 4 & \bar{5} & 0 & 4 & \bar{5} \\ 2 & 3 & 13 & 2 & 3 \\ 7 & \bar{4} & 10 & 7 & \bar{4} \end{array}\right.}{27} = \frac{120 - 455 + 0 - 0 + 208 + 100}{27} = \frac{-27}{27} = -1.$$

$\therefore\ x = 3,\ y = 2,$ and $z = -1.$

PROBLEMS

Solve the following equations by determinants. Check.

1. $3x - 4y + z = 14,$ $2x + 3y - 2z = -10,$ $5x - 2y + 3z = 18.$

2. $x + y - z = 1,$ $x - y + z = 3,$ $-x + y + z = 5.$

3. $6x + 3y - 4z = 2,$ $5x - 6y - 3z = -33,$ $2x - 5y + 2z = -30.$

4. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 13,$ $\frac{x}{3} - \frac{y}{4} + \frac{z}{2} = 7,$ $\frac{x}{4} + \frac{y}{2} - \frac{z}{3} = 5.$

5. $2x + y + z = 16,$ $x + 2y + z = 9,$ $x + y + 2z = 3.$

6. $x - 2y + 3z = 2,$ $2x - 3y + z = 1,$ $3x - y + 2z = 9.$

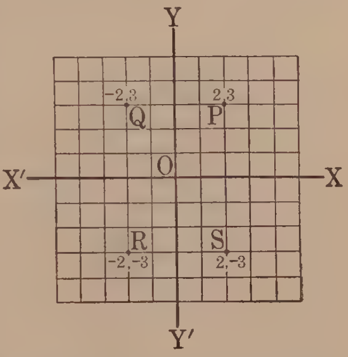
7. $5x + 2y = 14,$ $y - 6z = 3,$ $x + 2y + z = -3.$

8. $ax - by = c,$ $by + cz = a,$ $ax - cz = b.$

9. $3x - 4y = a,$ $3z - 2x = -3a,$ $5y + 4z = 14a.$

GRAPHS OF EQUATIONS

340. **Axes.**—As in chapter I, two straight lines perpendicular to each other are called **axes**, as XX_1 and YY_1, or simply the **x-axis** and the **y-axis**. XX_1 is called the axis of **abscissas** and YY_1 the axis of **ordinates.** O is called the origin. All positive values of **x** are measured to the right of the **y-axis** and all negative values of **x** are measured to the left of the **y-axis.** All positive values of **y** are measured above the **x-axis** and all negative values of **y** are measured below the **x-axis.**



Thus, the point (2, 3) means $x = 2$ and $y = 3$ and is plotted by measuring 2 units to the right and 3 units up. Read §1. Likewise (−2, 3) is plotted by measuring 2 units to the left and 3 units up; (−2, − 3), by measuring 2 units to the left and 3 units down; (2, −3), by measuring 2 units to the right and 3 units down as in the figure.

The **coordinates** of a point are the abscissa and ordinate taken together. They are written (2, 3). The first number is the **abscissa,** the second number is the **ordinate.**

The **quadrants** are the four parts into which the plane is divided by the axes. They are taken in order counter-clockwise. Thus P is in the first quadrant, Q in the second, R in the third, and S in the fourth.

PROBLEMS

Draw axes and locate the following points.

1. (4, 5), (−2, 4), (3, −3), (−1, −4), (−3, 6).
2. (0, 0), (−3, 0), (0, −2), (5, 0), (0, 3).
3. What is the ordinate of the x-axis?
4. What is the abscissa of the y-axis?
5. Plot each of the following points and find the distance between them. (2, 5) and (2, −5); (−3, 6) and (5, 6); (−4, −3) and (8, −3).
6. Plot the triangle whose vertices are (2, 2), (−3, 5), and (3, −5).
7. Plot the quadrilateral whose vertices are (3, 2), (−3, 2), (−3, −2), and (3, −2). What kind of a quadrilateral is this? Find its area.
8. Plot the rectangle whose vertices are (5, 4), (−2, 4), (−2, −3), and (5, −3). Find its area.
9. Plot the triangle whose vertices are (5, 1), (5, 4), and (1, 4). What kind of a triangle is this? Find its area.

341. **Plotting a Straight Line.**—Given the equation $2x - y = 3$.

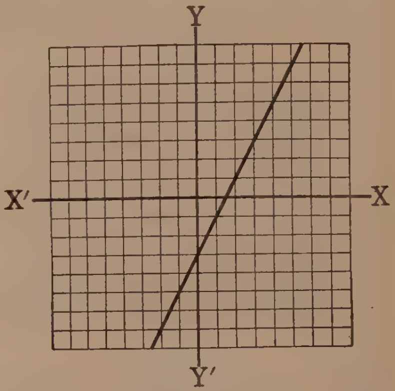
To construct its graph.

(*a*) Solve for y in terms of x. (*b*) Assign values to x, as 0, 1, 2, 3, etc. −1, −2, −3, etc. and find the corresponding values for y. Then locate the points which each pair of numbers determines.

Solution. $2x - y = 3$, $\therefore y = 2x - 3$.

Plot the points determined by the pairs of numbers shown in the accompanying table and connect the points. It is a straight line, and is the graph of $2x - y = 3$.

x	y
0	−3
1	−1
2	1
3	3
4	5
etc.	etc.
−1	−5
−2	−7
etc.	etc.



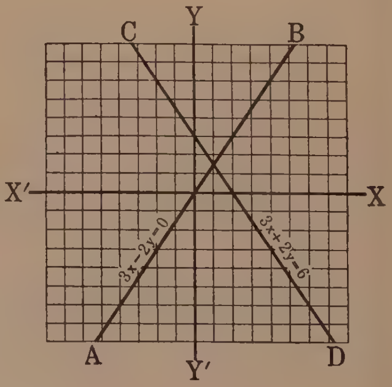
342. **Linear Equations.**—A linear equation is an equation whose graph is a straight line. It is proven in analytical geometry that the graph of every equation of the first degree in two variables is a straight line.

343. **To Construct the Graph of a Linear Equation by a Short Method (Intercepts).**—Two points determine a straight line (§90–2). Therefore we need to find only two points on the line to determine the line. To draw a graph of $3x + 2y = 6$, let $x = 0$ and solve for y, then let $y = 0$ and solve for x.

Thus, if $x = 0$, $y = 3$ and if $y = 0$, $x = 2$. Now plot the two points $(0, 3)$, and $(2, 0)$ and draw a straight line through them. Then CD is the graph of $3x + 2y = 6$.

It is obvious that the greater the distance between the two points chosen the more accurate the graph will be. However, if the graph passes through the origin it will be necessary to get one point in which neither x nor y is 0.

Thus, $3x - 2y = 0$. When $x = 0$, $y = 0$, but if $x = 2$, $y = 3$. Line AB on the figure is the graph of $3x - 2y = 0$.



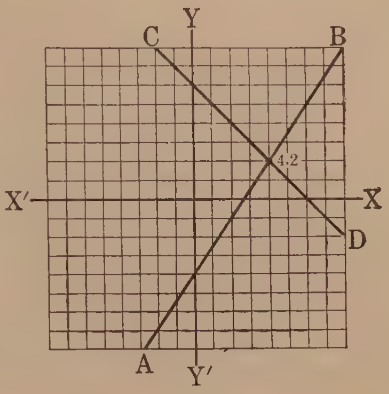
PROBLEMS

Plot the following linear equations.

1. $y = x + 3$.
2. $2x - 3y = 6$.
3. $4x + 5y = 10$.
4. $y = 3x$.
5. $3x - 5y - 15 = 0$.
6. $3x + 2y = 0$.
7. $\frac{x-2}{3} = 2y$.
8. $\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6}$.
9. $x = 3$.
10. When does the graph of a linear equation pass thru the origin? Does $5x + 2y = 0$ pass through the origin?

11. Plot the triangle whose sides are the graphs of, $2x - y = 1$, $x - 3y + 7 = 0$, and $3x + y = -11$.

344. Simultaneous Linear Equations Solved by Graphs.—If we draw the graph of the equations $3x - 2y = 8$ and $x + y = 6$ we get the lines *AB* and *CD* which intersect at the point (4, 2). If we solve the pair of simultaneous equations we get $x = 4$ and $y = 2$. Two lines will intersect at the point which will satisfy both equations for the values of x and y.



The graph is often used as an approximate check of an algebraic solution.

345. Independent Linear Equations.—Independent linear equations are equations which have a common solution. That means that each equation represents a different line, but that the two lines intersect. Thus, $3x - 2y = 8$ and $x + y = 6$ are independent.

346. Dependent Linear Equations.—Two linear equations are dependent when they have more than one solution. Thus, $x + y = 5$ and $2x + 2y = 10$ have all solutions in common. In fact they represent the same relation, for if $2x + 2y = 10$ is divided by 2 we have $x + y = 5$. Therefore when two equations are dependent they give the same graph. Solve the equations by algebra. Plot their graphs.

347. Inconsistent Linear Equations.—Two linear equations are inconsistent when they have no common solution. The graphs of inconsistent equations are parallel lines. Thus, $2x + 3y = 1$ and $2x + 3y = 3$ are inconsistent. Solve the equations by algebra. Plot their graphs.

PROBLEMS

Solve the following pairs of equations both graphically and algebraically and compare the results.

1. $2x - 3 = 0$, $y = 0$.
2. $3y + 4 = 0$, $x = 0$.
3. $2x - 7y = 3$, $3x + 2y = 17$.
4. $y = 2x + 5$, $x - 3y = -10$.
5. $x = 3y + 10$, $2y - 3x = -16$.
6. $3x - 2y = 6$, $9x - 6y = 4$.
7. $4x + 5y = 10$, $5y - 4x = 4$.
8. $2x + 5y = 10$, $8x + 20y = 5$.
9. $2x - 3y = 4$, $4x - 6y = 8$.
10. $2x + y = 5$, $4x + 2y = 13$.
11. $x + y = 5$, $x = 3$.
12. $x + 2y = 7$, $y = 3$.
13. $3x - y = 5$, $-6x + 2y = -10$.
14. $x + 2y = 3$, $2x + 4y = 2$.
15. Plot the triangle whose sides are $2x + 3y = 0$, $x + 3y = -3$, and $x + y = -1$. What are the coordinates of the vertices of this triangle?
16. Plot the parallelogram whose sides are $x - 5y = 0$, $5x - y + 17 = 0$, $x - 5y + 11 = 0$, and $5x - y - 7 = 0$. Find the coordinates of the vertices of the parallelogram.

CHAPTER X

POWERS, EXPONENTS, ROOTS, AND RADICALS

348. **Importance of Powers and Roots.**—Many of the formulas in later work in mathematics and in engineering subjects contain quantities involving powers and roots. A thorough knowledge of the contents of this chapter will facilitate the evaluation of these formulas.

349. **Power.**—A power is the product of two or more equal factors.

Thus, **a·a·a·a** means that a is to be used four times as a factor and the product is written a^4. In the same way $(a + b)(a + b)(a + b)$ equals $(a + b)^3$, and $b \cdot b \cdot b \; . \; . \; . \; b$ to **n** factors equals b^n.

350. **Involution.**—Involution is the process of finding the value of an expression raised to a given power.

351. **Laws of Signs.**—Following the law of signs in multiplication, tell the sign of each of the following: $(+a)^2$, $(+a)^3$, $(-a)^2$, $(+a)^5$, $(-a)^4$, $(-a)^5$.

What is the sign of an even power of a positive expression? Of an odd power? What is the sign of an even power of a negative expression? Of an odd power?

Hence, we have the following laws:

1. All powers of a positive expression are positive.
2. All even powers of a negative expression are positive.
3. All odd powers of a negative expression are negative.

352. **Exponents.**—Read again §15.

1. Find the value of 6^2; $(2 \times 3)^2$; $2^2 \times 3^2$. Compare results.

2. Find the value of 2^2; $\left(\frac{6}{3}\right)^2$; $\frac{6^2}{3^2}$. Compare results.

3. Find the value of $(3^2)^2$ and 3^4; $(4^3)^2$ and 4^6; $(a^2)^3$ and $(a^3)^2$. Compare the results in each pair.

From (1) we infer that $(ab)^n = a^nb^n$; from (2), that $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$; from (3) that $(a^m)^n = a^{mn}$.

353. **Laws of Exponents.**

1. Any power of a product is equal to the product of its factors each raised to the given power.

2. Any power of the quotient of two factors is equal to the quotient of the two factors each raised to the given power, *i.e.*, in multiplication and division the distributive law applies for exponents.

3. The required **power** of an expression, to a **given power,** is found by raising the expression to a power equal to the product of the two powers.

PROBLEMS

1. $2^2 \cdot 2^2$.
2. $3^2 \cdot 2^3$.
3. $(3 \cdot 4)^2$.
4. $(2^2)^2$.
5. $(-2^3)^2$.
6. $(2^2)^3$.
7. $(-4a^2b)^2$.
8. $(3mn)^3$.
9. $(2 \cdot 3 \cdot 4)^2$.
10. $(2xy)^n$.
11. $(-3a^2b)^{2n}$.
12. $(-pqr)^{2n-1}$.
13. $\left(\frac{3}{4}\right)^2$.
14. $\left(-\frac{3a^2b}{4ab^2}\right)^3$.
15. $\left(\frac{2a}{3b}\right)^n$.
16. $\left(-\frac{a}{b}\right)^{2n}$.
17. $\left(-\frac{a}{b}\right)^{2n-1}$.
18. $\left(-\frac{a}{b}\right)^{2n+1}$.
19. $\left[\frac{(-2)\,^3rs}{3xy}\right]^2$.
20. $\left(-\frac{3}{xy}\right)^3$.
21. $\left(-\frac{1}{2}\right)^3$.
22. $\left(-\frac{a^2b^3}{cd^2}\right)^3$.
23. $\left(\frac{1}{2}gt^2\right)^3$.
24. $\left(\frac{gt^2}{2}\right)^2$.
25. $\left(\frac{2x^n}{3y^m}\right)^r$.
26. $\left(\frac{-2x}{3a}\right)^2$.
27. $\left(\frac{3a}{2b^2}\right)^3$.

28. Find the value of $\left[2x - \frac{x^3}{6}\right]$,

(1) When $x = 2$. (2) When $x = 0$.

Subtract the result of (2) from the result of (1). *Ans.* $\frac{8}{3}$.

29. Find the value of $\frac{\pi r^2 x^4}{4h^2}$,

(1) When $x = h$. (2) When $x = 0$.

Subtract (2) from (1), as in problem 28. *Ans.* $\frac{\pi r^2 h^2}{4}$.

30. Find the value of $b\left(\frac{z^3}{3}\right)$,

(1) When $z = \frac{1}{2}d$. (2) When $z = -\frac{1}{2}d$.

Subtract (2) from (1). *Ans.* $\frac{1}{12}bd^3$.

31. Find the value of $\frac{\pi x^4}{2}$,

(1) When $x = \frac{b}{2}$. (2) When $x = \frac{d}{2}$.

Subtract (2) from (1). *Ans.* $\frac{\pi}{32}(b^4 - d^4)$.

32. Find the value of $\frac{b}{d}\left[\frac{2}{3}d \cdot \frac{z^3}{3} - \frac{z^4}{4}\right]$,

(1) When $z = \frac{2}{3}d$. (2) When $z = -\frac{1}{3}d$.

Subtract (2) from (1). *Ans.* $\frac{1}{36}bd^3$.

354. **Roots.**—Each of the equal factors of a number is a root of that number. If a number has two equal factors, one of the equal factors is the **square root** of that number. If there are three equal factors, one of them is the **cube root.**

355. **Evolution.**—Evolution is the process of finding the root of a number.

Illustrations. $25 = 5 \cdot 5 \therefore 5$ is the square root of 25.

$25 = (-5) \cdot (-5) \therefore -5$ is the square root of 25.

$289 = 17 \cdot 17 \therefore 17$ is the square root of 289.

$289 = (-17)(-17) \therefore -17$ is the square root of 289.

$-8 = (-2)(-2)(-2) \therefore -2$ is the cube root of -8.

$a^3 = a \cdot a \cdot a \therefore a$ is the cube root of a^3.

$a^6 = a^2 \cdot a^2 \cdot a^2 \therefore a^2$ is the cube root of a^6.

$a^5 = a \cdot a \cdot a \cdot a \cdot a \therefore a$ is the fifth root of a^5.

$(x + y)^4 = (x + y)(x + y)(x + y)(x + y) \therefore (x + y)$ is the fourth root of $(x + y)^4$.

356. **Principal Root.**—It is seen from the above illustrations that a positive number has two square roots, the one positive and the other negative. To indicate the two values, the double sign ± is used. Thus the square root of 25 is ± 5. The positive root is called the **principal root.** Every positive number has only one positive (**principal**) root.

It is also seen that the cube root of -8 is -2. There is no positive cube root of -8. The **principal** cube root of -8 is then -2. Every negative number has only one negative (**principal**) odd root.

It can be shown that any number has two square roots, three cube roots, or in general **n, n**th roots.

In most computations the **principal root** is the only root desired.

PROBLEMS

1. Find the two equal factors of the following: 4; a^2; 16; $16a^4b^2c^2$; $m^2 + 2mn + n^2$; $16(x - y)^2$; $36(r + s)^2$; 2; 3; -4; -25? Do 2, 3, -4, and -25 have two equal factors?

2. Compare the value of $+2 \cdot +2$ and $-2 \cdot -2$; $(m + n) \cdot (m + n)$ and $-(m + n) \cdot -(m + n)$. What is the sign of the square root of an expression? Does a negative number have two equal factors? Why?

3. Find the three equal factors of the following: 8; -8; $8a^6b^3$; $-8a^6n^3$; $27(x + y)^3$; $(a + b)^3$; $(m + n)^3$.

Note that the cube root of an expression has the same sign as the expression.

4. Find the four equal factors of 16; $81a^4b^4$; $16x^4y^8$.

357. **Zero and Negative Exponents.**—Read again §§46 and 47. When an exponent is mentioned we think of it as telling how many times an expression is to be used as a factor. Thus a^3 means **a** is taken three times as a factor. In division, however, it was found that by dividing $\mathbf{a}^n$ by $\mathbf{a}^n$ and following the law of exponents the result was a^0. We found that the value of $a^0 = 1$.

Again, 1. $a^3 \div a^5 = a^{3-5}$ or a^{-2}.

2. $a^5b^3 \div a^7b^4 = a^{-2}b^{-1}$.

3. $(a + b)^2 \div (a + b)^3 = (a + b)^{-1}$.

358. **Negative Exponents Eliminated.**—Negative exponents are awkward to handle in performing algebraic operations. They can be manipulated in such a way as to change an expression having a negative exponent to an equivalent one having a positive exponent.

Illustrations.—Change the following expressions to equivalent expressions having positive exponents.

(1) a^{-2}. (2) $-a^2b^{-3}$. (3) $\frac{a^{-2}}{b^{-3}}$. (4) $2(a^2 - x^2)^{-1}$.

Solution.—(1) $$a^{-2} \times \frac{a^2}{a^2} = \frac{a^0}{a^2} = \frac{1}{a^2}.$$

(2) $$-a^2b^{-3} \times \frac{b^3}{b^3} = -\frac{a^2b^0}{b^3} = -\frac{a^2}{b^3}.$$

(3) $$\frac{a^{-2}}{b^{-3}} \times \frac{a^2b^3}{a^2b^3} = \frac{a^0b^3}{a^2b^0} = \frac{b^3}{a^2}.$$

(4) $$2(a^2 - x^2)^{-1} \times \frac{(a^2 - x^2)}{(a^2 - x^2)} = \frac{2(a^2 - x^2)^0}{a^2 - x^2} = \frac{2}{a^2 - x^2}.$$

Hence, a **factor** *may be changed from numerator to denominator or from denominator to numerator, provided the sign of the exponent is changed.* **Note** *that these transfers have nothing to do with the sign of the expression itself as shown in illustration* 2.

PROBLEMS

Write the following with positive exponents:

1. 2^{-3}.
2. 3^{-2}.
3. ab^{-1}.
4. $-a^{-2}b$
5. $3x^{-2}$.
6. $3 \cdot 2^{-1}x^2$.
7. $2^{-1}x^2$.
8. x^3y^{-2}.
9. $(a^2 - x^2)^{-1}$.
10. $-2(a + b)^{-2}$.
11. $3^{-2}(x^2 - y^2)$.
12. $-2(a^2 - b^2)^{-2}$.
13. $\frac{a^{-2}}{c^{-3}}$.
14. $\frac{2b^{-1}}{b^{-2}}$.
15. $\frac{ab^{-1}c}{2^{-2}}$.
16. $\frac{3(x - y)^{-1}}{x + y}$.
17. $\left(\frac{a + b}{a - b}\right)^{-2}$.
18. $\left(\frac{x^2 - y^2}{x + y}\right)^{-1}$.
19. $\left(\frac{r - s}{p + q}\right)^{-3}$.
20. $\frac{2^{-3}}{3^{-2}}$.

359. **Fractional Exponents.**—By §18–1, we have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}$ equals **a**. It follows that $a^{\frac{1}{2}}$ is the square root of **a**. What exponent denotes the square root of an expression?

Again, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a$. Hence $a^{\frac{1}{3}}$ is the cube root of **a**. What exponent denotes the cube root of an expression? In general $a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}}$ to n factors $= a$. What exponent denotes the nth root of an expression?

By §353–3, we have $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = (a^{\frac{1}{3}})^2 = a^{\frac{2}{3}}$.

Again, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = (a^{\frac{1}{2}})^3 = a^{\frac{3}{2}}$.

When $a^{\frac{1}{3}}$ is used twice as a factor we have $a^{\frac{2}{3}}$. When $a^{\frac{1}{2}}$ is used three times as a factor we have $a^{\frac{3}{2}}$.

Hence, in a fractional exponent of an expression the numerator denotes the power and the denominator denotes the root of that expression.

360. **Summary of Exponents.**

Exponents	Integral	Positive or Negative.
	Fractional	Positive or Negative.
	Zero	

361. Multiplication and Division of Expressions Containing Fractional Exponents.—To multiply or divide expressions having fractional exponents with different denominators, follow the laws of addition and subtraction of fractions in adding and subtracting the exponents.

Illustrations.

1. $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}$.
2. $a^{\frac{2}{3}} \div a^{\frac{1}{4}} = a^{\frac{2}{3}-\frac{1}{4}} = a^{\frac{5}{12}}$.
3. $(m + n)^{\frac{1}{3}}(m + n)^{\frac{1}{4}} = (m + n)^{\frac{1}{3}+\frac{1}{4}} = (m + n)^{\frac{7}{12}}$.
4. $(p - q)^{\frac{3}{4}} \div (p - q)^{\frac{2}{3}} = (p - q)^{\frac{3}{4}-\frac{2}{3}} = (p - q)^{\frac{1}{12}}$.

PROBLEMS

Find the value of the following:

1. $9^{\frac{1}{2}}$. 2. $9^{\frac{3}{2}}$. 3. $9^{\frac{5}{2}}$. 4. $27^{\frac{1}{3}}$. 5. $27^{\frac{2}{3}}$.
6. $27^{\frac{3}{3}}$. 7. $(-27)^{\frac{1}{3}}$. 8. $(-27)^{\frac{2}{3}}$. 9. $(-27)^{-\frac{1}{3}}$. 10. $81^{\frac{3}{4}}$.
11. $(a^{-2})^3$. 12. $(a^{-\frac{1}{2}}b^{\frac{1}{3}})^3$. 13. $100^{-\frac{3}{2}}$. 14. $(y^{\frac{3}{2}})^2$. 15. $4^{\frac{1}{2}} \times 36^{\frac{1}{2}}$.
16. $4^{\frac{3}{2}} + 8^{\frac{2}{3}}$. 17. $\frac{1}{4^{\frac{1}{2}}} + \frac{1}{8^{\frac{1}{3}}}$. 18. $\frac{1}{4^{-\frac{3}{2}}} + \frac{1}{8^{-\frac{2}{3}}}$. 19. $9^{-\frac{3}{2}} \times 27^{\frac{2}{3}}$.

20. $\frac{(x + y)^2}{(x - y)} \frac{(x - y)^{-3}}{(x + y)}$. 21. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$.
22. $(a^2 - b^2) \div (a^2 - b^2)^{\frac{1}{2}}$. 23. $(a^2 - b^2)^{\frac{1}{2}} \div (a^2 - b^2)^{\frac{1}{3}}$.
24. $2a(a - b)^{-\frac{1}{2}}$. 25. $\frac{mn}{(p + q)^{-2}}$.
26. Simplify $\frac{2}{x^{\frac{1}{2}} - 1} - \frac{2x^{\frac{1}{2}}}{x - 1} + \frac{1}{x^{\frac{1}{2}} + 1}$.

Find the value of the following, using the given conditions. Every example should be solved.

27. $(1 + 4x)^{\frac{1}{2}}$, when $x = 2$. *Ans.* ± 3.
28. $(6 + 7x)^{\frac{2}{3}}$, when $x = 3$. *Ans.* 9.
29. $(3\frac{2}{9} + \frac{4}{9}x)^{\frac{1}{2}}$, when $x = 5$. *Ans.* $\pm \frac{7}{3}$.
30. $(1 + \frac{2}{3}x)^{\frac{3}{2}}$, when $x = 12$. *Ans.* ± 27.
31. $(8x^2 + 15x + 8\frac{1}{2})^{\frac{1}{2}}$, when $x = \frac{3}{2}$. *Ans.* ± 7.
32. $\frac{8\pi}{3k} \cdot \frac{r^4}{(5r^2 - a^2)^{\frac{1}{2}}}$, when $r = a$. *Ans.* $\pm \frac{4\pi a^3}{3k}$.
33. $\frac{x}{a^2(8x^2 + a^2)^{\frac{1}{2}}}$, when $x = a$. *Ans.* $\pm \frac{1}{3a^2}$.
34. $\frac{(3x^2 - 3a^2)^{\frac{n+2}{2}}}{n + 2}$, when $x = 2a$ and $n = 1$. *Ans.* $\pm 9a^3$.
35. $\frac{-(2ax - x^2)^{\frac{n+2}{n+1}}}{3ax^3}$, when $x = a$ and $n = 1$. *Ans.* $-\frac{1}{3a}$.

362. The Radical or Root Sign.—The sign $\sqrt{\ }$ written over an expression indicates the square root of the expression. This sign is made up of the letter **r** (root) and the vinculum. When a root, other than the square root, is desired it is necessary to place in the **r** a number called the index number. Thus, $\sqrt[3]{\ }$ indicates cube root, $\sqrt[4]{\ }$ indicates fourth root, etc. Read the following:

$$\sqrt{4},\ \sqrt{8},\ \sqrt[4]{16},\ \sqrt[3]{1728},\ \sqrt[3]{a^3b^3}.$$

The index number indicates the **order** of the expression.

363. **Rational Expressions.**—A rational expression is one of which the indicated root can be taken, thus, $\sqrt{16} = \pm 4$ and $\sqrt{25} = \pm 5$.

364. **Irrational Expression.**—An irrational expression is one of which the indicated root cannot be taken as $\sqrt{5}$, $\sqrt{17}$, $\sqrt{x^2 + xy + y^2}$. These expressions are also called **radicals** or **surds.**

365. **Relation between Radicals and Fractional Exponents.**—The student must continually compare the radical expression with expressions having fractional exponents.

Thus, $\sqrt{2} = 2^{\frac{1}{2}}$; $\sqrt{a^2 - x^2} = (a^2 - x^2)^{\frac{1}{2}}$; $\sqrt[3]{(a^2 - x^2)^2} = (a^2 - x^2)^{\frac{2}{3}}$.

PROBLEMS

Express the following with fractional exponents:

1. $\sqrt{8}$. 2. $\sqrt{a + b}$. 3. $\sqrt[3]{(a^2 + b^2)^2}$. 4. $\sqrt{(x^2 - y^2)^3}$.

5. $\sqrt{x^2y^3}$. 6. $\sqrt[3]{p^2q^4}$. 7. $\sqrt{25a^3}$. 8. $\sqrt[3]{8b^2}$.

9. $\sqrt{\dfrac{x^3}{y^2}}$. 10. $\sqrt[3]{\dfrac{m^2n^3}{k^2}}$. 11. $\sqrt{\dfrac{a + b}{a + b}}$. 12. $\sqrt{\dfrac{m - n}{p + q}}$.

366. **Roots of Expressions Which Can be Factored.**—Compare the following. Give your conclusion in every case.

1. $\sqrt{36}$, $\sqrt{4 \times 9}$, and $\sqrt{4} \cdot \sqrt{9}$. 2. $\sqrt{a^4b^2}$ and $\sqrt{a^4} \cdot \sqrt{b^2}$.

3. $\sqrt[3]{216}$ and $\sqrt[3]{8} \cdot \sqrt[3]{27}$. 4. $\sqrt[3]{a^6b^9c^3}$ and $\sqrt[3]{a^6} \cdot \sqrt[3]{b^9} \cdot \sqrt[3]{c^3}$.

Hence, the root of any expression is equal to the product of the same root of its factors. In (4), *we see that we may divide the power by the index to obtain the result.*

367. **Roots of Fractions.**—Compare the following. Give your conclusion.

1. $\sqrt{\dfrac{16}{4}}$ and $\dfrac{\sqrt{16}}{\sqrt{4}}$. 2. $\sqrt[3]{\dfrac{a^4}{a^2}}$ and $\dfrac{\sqrt[3]{a^4}}{\sqrt[3]{a^2}}$. 3. $\sqrt{\dfrac{(a + b)}{(m - n)}}$ and $\dfrac{\sqrt{a + b}}{\sqrt{m - n}}$.

Hence, the root of a fraction is equal to the same root of the numerator, divided by the same root of the denominator.

368. **Incommensurable Numbers.**—Numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., are incommensurable because their value can be determined only approximately. Although we can give only an approximate numerical value to these numbers they can be constructed as accurately as any other number.

It will be remembered from arithmetic that in any right triangle the sum of the squares of the legs is equal to the square of the hypotenuse. This theorem is proved in §411.

Illustrations.—1. Construct $\sqrt{2}$.

Solution.—Construct an isosceles right triangle with each leg one unit in length. The hypotenuse will be the $\sqrt{2}$.

2. Construct $\sqrt{3}$. Construct a right triangle with legs $\sqrt{2}$ and 1. The hypotenuse is $\sqrt{3}$.

PROBLEMS

Construct:

$$\sqrt{7},\ \sqrt{5},\ \sqrt{17},\ \sqrt{11},\ \sqrt{53},\ \sqrt{41},\ \sqrt{6},\ \sqrt{15},\ \sqrt{10},\ \sqrt{19}.$$

369. **Imaginary Numbers.**—An imaginary number is an indicated even root of a negative number. Because of the law of signs in multiplication, we cannot find a number which when multiplied by itself will give a negative number. We cannot find any expression which when used an even number of times as a factor gives a negative result.

Hence, an even root of a negative number is impossible.

Expressions such as $\sqrt{-4}$, $\sqrt{-25}$, $\sqrt{-a^2b^2}$ often occur in mathematical work and must be dealt with. In order to distinguish them from all other expressions (real) we call them imaginary. The imaginary unit is $\sqrt{-1}$, often written i. Any expression of which an even root of a negative quantity is required can be reduced to this form.

Thus, $\sqrt{-4} = \sqrt{4(-1)} = \pm 2\sqrt{-1}$ or $\pm 2\,i$.

370. **Powers of $\sqrt{-1}$.**

$$\begin{aligned}
i &= \sqrt{-1} = i.\\
i^2 &= (\sqrt{-1})^2 = \sqrt{-1}\cdot\sqrt{-1} = -1.\\
i^3 &= (\sqrt{-1})^3 = (\sqrt{-1})^2\cdot\sqrt{-1} = -\sqrt{-1} = -i.\\
i^4 &= (\sqrt{-1})^4 = (\sqrt{-1})^2\cdot(\sqrt{-1})^2 = 1.\\
i^5 &= (\sqrt{-1})^5 = (\sqrt{-1})^4\cdot\sqrt{-1} = \sqrt{-1} = i.
\end{aligned}$$

Hence, it can be shown that the powers of $\sqrt{-1}$ have the values i, -1, $-i$ and 1.

The powers of i can be shown graphically if desired.

371. **Summary.**—1. An even root of a positive quantity is either positive or negative.

2. An odd root of a negative quantity is negative.

3. The root of an expression is equal to the product of the same root of its factors.

4. An incommensurable number can be represented graphically.

5. An even root of a negative quantity cannot be taken and is imaginary.

372. **Multiplication of Polynomials Containing Radicals.**—To multiply polynomials containing radicals follow the general method of multiplication.

Illustration.—Multiply $2\sqrt{3} + 3\sqrt{2}$ by $3\sqrt{3} - 2\sqrt{2}$.

Solution.

$$\begin{array}{rrr}
2\sqrt{3} + 3\sqrt{2} & & \\
3\sqrt{3} - 2\sqrt{2} & & \\
\hline
18 + 9\sqrt{6} & & \\
-\ 4\sqrt{6} & - 12 & \\
\hline
18 + 5\sqrt{6} & - 12 & \text{or } 6 + 5\sqrt{6}.
\end{array}$$

PROBLEMS

Find the value of the following:

1. $(2 + \sqrt{3})(2 - \sqrt{3})$.
2. $(3 - \sqrt{2})(3 + \sqrt{2})$.
3. $(a + 2\sqrt{3})(b - 3\sqrt{2})$.
4. $(m - \sqrt{n})(n + \sqrt{m})$.

373. **Radicals Simplified.**—A radical is in its simplest form when there is no factor under the radical sign of which the indicated root can be taken, and when there is no radical in the denominator.

Illustrations.—1. Simplify $\sqrt{12}$.

Solution. $\sqrt{12} = \sqrt{4 \cdot 3} = \pm 2\sqrt{3}$.

2. Simplify $\frac{1}{\sqrt{2}}$.

Solution.—Multiply both numerator and denominator by $\sqrt{2}$.

Thus, $\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ or $\frac{1}{2}\sqrt{2}$.

3. Simplify $\sqrt{\frac{2}{3}}$.

Solution. $\frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ or $\frac{1}{3}\sqrt{6}$.

4. Simplify $\frac{2}{\sqrt{3} + \sqrt{5}}$.

Solution.—Multiply both numerator and denominator by $\sqrt{3} - \sqrt{5}$.

Note.—*The expression,* $\sqrt{3} - \sqrt{5}$, *is called the conjugate of* $\sqrt{3} + \sqrt{5}$.

Thus, $\frac{2}{\sqrt{3} + \sqrt{5}} \times \frac{\sqrt{3} - \sqrt{5}}{\sqrt{3} - \sqrt{5}} = \frac{2(\sqrt{3} - \sqrt{5})}{3 - 5} = \sqrt{5} - \sqrt{3}$.

PROBLEMS

Simplify the following:

1. $\sqrt{20}$.
2. $\sqrt{72}$.
3. $\sqrt{216}$.
4. $\sqrt{125}$.
5. $\frac{1}{\sqrt{3}}$.
6. $\frac{2}{\sqrt{5}}$.
7. $\frac{3}{\sqrt{3}}$.
8. $\frac{2}{\sqrt{2}}$.
9. $\sqrt{\frac{3}{4}}$.
10. $\sqrt{\frac{5}{8}}$.
11. $\sqrt{\frac{8}{2}}$.
12. $\sqrt{\frac{9}{5}}$.
13. $\frac{1}{\sqrt{3} - \sqrt{2}}$.
14. $\frac{2}{3 + \sqrt{5}}$.
15. $\frac{a}{\sqrt{b} - \sqrt{c}}$.

374. **Addition and Subtraction of Radicals.**—To find the algebraic sum of two or more radicals, reduce each one to its simplest form and collect similar radicals.

Illustrations.

1. Simplify $\sqrt{18} + \sqrt{12} - \sqrt{8}$.

Solution. $3\sqrt{2} + 2\sqrt{3} - 2\sqrt{2} = \sqrt{2} + 2\sqrt{3}$.

2. Simplify $2\sqrt{a^2bx} + a\sqrt{bx} - \sqrt{4b^3x}$.

Solution. $2a\sqrt{bx} + a\sqrt{bx} - 2b\sqrt{bx} = 3a\sqrt{bx} - 2b\sqrt{bx}$ or $(3a - 2b)\sqrt{bx}$.

375. Multiplication and Division of Radicals.—Before radicals can be multiplied or divided they must be reduced to radicals of the same order. (See §362.)

Illustrations.—1. Multiply $\sqrt{2}$ by $\sqrt[3]{3}$.

Solution. $\sqrt{2} = 2^{\frac{1}{2}}$ and $\sqrt[3]{3} = 3^{\frac{1}{3}}$.

Hence, $2^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \cdot 3^{\frac{2}{6}} = (2^3)^{\frac{1}{6}} \cdot (3^2)^{\frac{1}{6}} = (8 \cdot 9)^{\frac{1}{6}} = (72)^{\frac{1}{6}} = \sqrt[6]{72}$.

2. Divide $\sqrt{21}$ by $\sqrt{3}$.

Solution. $\sqrt{21} \div \sqrt{3} = \sqrt{\frac{21}{3}} = \sqrt{7}$.

3. Divide $14\sqrt{2}$ by $2\sqrt[3]{6}$.

Solution. $$14\sqrt{2} \div 2\sqrt[3]{6} = \frac{14\sqrt{2}}{2\sqrt[3]{6}} = 7 \cdot \frac{2^{\frac{1}{2}}}{6^{\frac{1}{3}}} = 7 \cdot \frac{2^{\frac{3}{6}}}{6^{\frac{2}{6}}} = 7\left(\frac{2^3}{6^2}\right)^{\frac{1}{6}} =$$
$$7\sqrt[6]{\tfrac{2}{9}} = 7\sqrt[6]{\tfrac{2}{9} \times \tfrac{81}{81}} = \tfrac{7}{3}\sqrt[6]{162}.$$

PROBLEMS

Simplify the following:

1. $2\sqrt{2} + \sqrt{18} - \sqrt{8}$.
2. $\sqrt{45} - \sqrt{5}$.
3. $\sqrt{72} + 2\sqrt{8}$.
4. $2\sqrt{3} - \sqrt{48}$.
5. $3\sqrt{27} - \sqrt{75}$.
6. $3\sqrt[3]{16} + 2\sqrt[3]{54}$.
7. $\sqrt[3]{625} - 3\sqrt[3]{135}$.
8. $2\sqrt{50} - 2\sqrt{32}$.
9. $\sqrt{2} - \dfrac{1}{\sqrt{2}}$.
10. $\sqrt{3} + \sqrt{\frac{1}{3}}$.
11. $\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}}$.
12. $\sqrt{\frac{3}{2}} - \sqrt{\frac{2}{3}}$.
13. $\sqrt{3} \times \sqrt{12}$.
14. $2\sqrt{12} \times 3\sqrt{18}$.
15. $\sqrt{12} \times \sqrt{27}$.
16. $\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{3}}$.
17. $\sqrt{\frac{3}{4}} \times \sqrt{\frac{8}{3}}$.
18. $\sqrt{\frac{5}{6}} \times \sqrt{\frac{18}{10}}$.
19. $3\sqrt{a^3} \times 2\sqrt{a}$.
20. $\sqrt[3]{2b^3} \times \sqrt[3]{16}$.
21. $\sqrt{32} \div \sqrt{8}$.
22. $\sqrt{27} \div \sqrt{3}$.
23. $\sqrt{6} \div \sqrt{18}$.
24. $3\sqrt{60} \div 4\sqrt{5}$.
25. $2\sqrt{\frac{5}{6}} \div \sqrt{\frac{15}{32}}$.
26. $6\sqrt{\frac{2}{3}} \div 3\sqrt{\frac{1}{12}}$.
27. $4\sqrt{\frac{3}{2}} \div \sqrt{\frac{5}{12}}$.
28. $\sqrt[3]{3} \div \sqrt{2}$.
29. $\dfrac{2}{\sqrt{2} - \sqrt{3}}$.
30. $\dfrac{5}{\sqrt{3} + \sqrt{5}}$.
31. $\dfrac{3}{5 + \sqrt{3}}$.
32. $\dfrac{4}{2\sqrt{3} - 3\sqrt{2}}$.
33. $\dfrac{\sqrt{a}}{\sqrt{a} \div \sqrt{b}}$.

Which of the following statements are true? Prove your answer.

35. $\dfrac{2}{\sqrt{2}} = \sqrt{2}$.
36. $\dfrac{1}{\sqrt{2}} = \dfrac{1}{2}\sqrt{2}$.
37. $\sqrt{\dfrac{2 \times 3}{4}} = \sqrt{\dfrac{2 \times 3 \times 4}{4}}$.
38. $\sqrt{\dfrac{xy}{pq}} = \sqrt{\dfrac{xypq}{xy}}$.
39. $\sqrt{4} + \sqrt{9} = \sqrt{4 + 9}$.
40. $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$.
41. $\sqrt{x^2 + y^2} = x + y$.
42. $\sqrt{a^2 + ab + b^2} = a + b$.
43. $\sqrt{m^2 + 2mn + n^2} = m + n$.
44. $\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c} = \sqrt{abc}$.

376. Equations Involving Fractional Exponents and Radicals.

Illustration.—1. Solve $x^{\frac{1}{2}} = 4$.

Solution.—Squaring both sides of the equation,

$$(x^{\frac{1}{2}})^2 = 4^2 \quad \therefore x = 16.$$

Check. $(16)^{\frac{1}{2}} = 4$ or $4 \equiv 4$.

2. Solve $x^{\frac{2}{3}} = 9$.

Solution.—Raising both sides of the equation to the ^{3} power,

$$(x^{\frac{2}{3}})^{\frac{3}{2}} = 9^{\frac{3}{2}} \quad \therefore x = 27.$$

Check. $(27)^{\frac{2}{3}} = 9$ or $9 \equiv 9$.

3. Solve $\sqrt{x+7} = 16$.

Solution.—Squaring both sides of the equation,

$$x + 7 = 256 \quad \therefore x = 249.$$

Check. $\sqrt{249 + 7} = 16$ or $16 \equiv 16$.

4. Solve $\sqrt{x^2 - 3} - x = 2$.

Solution.—Transposing, $\sqrt{x^2 - 3} = 2 + x$.

Squaring, $\qquad x^2 - 3 = 4 + 4x + x^2$.

$$\therefore \quad -7 = 4x \text{ or } x = -\tfrac{7}{4}.$$

Check. $\sqrt{\tfrac{49}{16} - \tfrac{48}{16}} + \tfrac{7}{4} = 2. \qquad \tfrac{1}{4} + \tfrac{7}{4} = 2$ or $2 \equiv 2$.

PROBLEMS

Solve the following and test:

1. $x^{\frac{1}{2}} = 15$. 2. $y^{\frac{1}{3}} = 6$. 3. $x^{\frac{3}{2}} = 27$. 4. $x^{\frac{2}{3}} = 9$.
5. $r^{\frac{3}{4}} = 8$. 6. $x^{\frac{1}{4}} = 3$. 7. $x^{-\frac{1}{2}} = 3$. 8. $v^{-\frac{2}{3}} = 4$.
9. $x^{-\frac{3}{2}} = 8$. 10. $n^{-2} = \frac{1}{36}$. 11. $\sqrt{x+1} = 9$. 12. $\sqrt{y-3} = 12$.
13. $\sqrt[3]{x-3} = 2$. 14. $\sqrt{2x-3} = 5$. 15. $\sqrt{x+2} - \sqrt{x} = 3$.
16. $\sqrt{x^2+3} = 1 - x$. 17. $\sqrt{x^2-5} - x = 5$. 18. $\sqrt{x^2-5} + x = 3$.
19. $\sqrt{x^2+7} = x + \frac{7}{3}$. 20. $\sqrt{x^2+3} = \frac{5}{2} - x$.

Problems involving **fractional and negative exponents** of the type which occur in Calculus. This list must be mastered by every student who expects to study Calculus.

Simplify.

21. $\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)$. *Ans.* $-\dfrac{x}{\sqrt{a^2 - x^2}}$.

22. $2 + (4r^2 - x^2)^{-\frac{1}{2}}(-2x)$. *Ans.* $\dfrac{2(\sqrt{4r^2 - x^2} - x)}{\sqrt{4r^2 - x^2}}$

23. $\frac{\pi x}{2}(4r^2 - x^2)^{-\frac{1}{2}}(-2x)$. *Ans.* $-\dfrac{\pi x^2}{\sqrt{4r^2 - x^2}}$.

24. $x^3 \cdot \frac{1}{2}(144 - x^2)^{-\frac{1}{2}}(-2x) + (144 - x^2)^{\frac{1}{2}}(3x^2)$. *Ans.* $\dfrac{4x^2(108 - x^2)}{\sqrt{144 - x^2}}$.

25. $(a^2x + x^3)\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x) + (a^2 - x^2)^{\frac{1}{2}}(a^2 + 3x^2)$.

Ans. $\dfrac{a^4 + a^2x^2 - 4x^4}{\sqrt{a^2 - x^2}}$.

26. $\dfrac{(a^2 - x^2)^{\frac{1}{2}}(2x) - (a^2 + x^2)\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)}{(a^2 - x^2)}$.

Hint.—*Multiply numerator and denominator by* $(a^2 - x^2)^{\frac{1}{2}}$ *and remember that any expression to the* 0 *power is* 1.

Ans. $\dfrac{3a^2x - x^3}{(a^2 - x^2)^{\frac{3}{2}}}$.

27. $\frac{1}{2}\left[x \cdot \frac{1}{2}(h^2 - x^2)^{-\frac{1}{2}}(-2x) + (h^2 - x^2)^{\frac{1}{2}}\right]$. *Ans.* $\dfrac{h^2 - 2x^2}{2(h^2 - x^2)^{\frac{1}{2}}}$.

28. $\dfrac{(100 + x^2)^{\frac{3}{2}} \cdot k - kx \cdot \frac{3}{2}(100 + x^2)^{\frac{1}{2}}(2x)}{(100 + x^2)^3}$. *Ans.* $\dfrac{2k(50 - x^2)}{(100 + x^2)^{\frac{5}{2}}}$

29. $\frac{1}{2}(2px)^{-\frac{1}{2}}(2p)$. *Ans.* $\dfrac{p}{\sqrt{2px}}$.

30. $\dfrac{\frac{1}{a}}{\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}}$. *Ans.* $\dfrac{1}{\sqrt{a^2 - x^2}}$.

31. $\dfrac{\frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)}{(1 - x^2)^{\frac{1}{2}}}$. *Ans.* $\dfrac{x}{x^2 - 1}$.

Find the value of the following under the given conditions.

32. $\dfrac{-2\sqrt{2ax - x^2}}{ax}$, when $x = a$. *Ans.* $-\dfrac{2}{a}$.

33. $-\dfrac{x + 3a}{2}\sqrt{2ax - x^2}$, when $x = a$. *Ans.* $-2a^2$.

34. $-\dfrac{1}{3}(9 - x^2)^{\frac{3}{2}}$, when $x = 3$, $x = 0$. Subtract the results. *Ans.* 9.

35. $\dfrac{x}{a^2\sqrt{a^2 + x^2}}$, when $x = a$. *Ans.* $\dfrac{\sqrt{2}}{2a^2}$.

36. $-\sqrt{\dfrac{x^2 + a^2}{a^2x}}$, when $x = 2a$. *Ans.* $-\dfrac{\sqrt{10a}}{2a}$.

37. $\dfrac{x(x^2 - a^2)^{\frac{n}{2}}}{n + 1}$, when $x = 2a$ and $n = 1$. *Ans.* $a^2\sqrt{3}$.

38. $\dfrac{(x^2 - a^2)^{\frac{n+2}{2}}}{n + 2}$, when $x = 3a$ and $n = 1$. *Ans.* $\dfrac{16a^3\sqrt{2}}{3}$.

377. **The Binomial Theorem.**—By actual multiplication we have,

1. $(a + b)^2 = a^2 + 2ab + b^2$.
2. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
3. $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
4. $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

A careful study of the above results will reveal some interesting and important facts on the expansion of a binomial.

In the above binomial **a** is the first term and **b** the second term. Let **n** represent the value of the successive powers. In the above $n = 2$, 3, etc.

378. **Laws for the Expansion of a Binomial.**—1. The first term of the expansion is a^n, the last term is b^n.

2. The number of terms is $n + 1$.

3. The sum of the exponents of **a** and **b** in every term is **n**.

4. The exponent of **a** decreases by 1 from **n**, in the first term, to **0**, in the last term ($a^0 = 1$).

5. The exponent of **b** increases by 1 from **0**, in the first term, to **n**, in the last term.

6. The coefficient of the first term is 1. The coefficient of the second term is **n**. The coefficient of any succeeding term is found by multiplying the coefficient of the preceding term by the exponent of **a** in that term and dividing the result by one more than the exponent of **b** in that term. Thus, the coefficient for the **third** term in the expansion of $(a + b)^4$ is $\dfrac{4 \times 3}{2}$ or 6.

7. All signs are plus.

If we find various powers of $(a - b)$, it will be seen that the first six laws are true. Number 7 will be replaced as follows:

8. If the binomial is the difference of two terms the signs of the expansion will be alternately plus and minus. This will be evident if (b) is replaced by $(-b)$ in §377–2.

Thus, $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Illustration.—Expand $(a + b)^5$.

Solution.

$$(a + b)^5 = a^5b^0 + \frac{5a^4b}{1} + \frac{5\cdot4}{1\cdot2}a^3b^2 + \frac{5\cdot4\cdot3}{1\cdot2\cdot3}a^2b^3 + \frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}ab^4 + \frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5}a^0b^5.$$

Note 1.—*The next coefficient will be* 0, *hence the expansion is complete.*

Note 2.—1·2·3·4 *is also written* $\underline{|4}$ *or* 4! *and is read factorial* 4.

When this expansion is simplified we obtain the same result as in §377–4.

Note.—*Observe that the number of factors in the numerator and in the denominator of each coefficient is* 1 *less than the number of the term, hence the coefficient of the* $(k + 1)^{th}$ *term has* k *factors in the numerator and in the denominator. The last factor in the numerator is* $n - (k - 1)$ *and the last factor in the denominator is* k.

Illustration.—In $(a + b)^5$, let $k + 1 = 4$.

Consider the 4th term. There are 3 factors in the numerator.

The last factor is 3. Hence it is $n - (k - 1)$.

Substituting, $5 - (3 - 1) = 3$.

There are 3 factors in the denominator 1·2·3. The last one is k or 3.

The exponent of **a** is **n** − **k** and the exponent of **b** is **k.**

379. **Induction.**—Induction is the process of proving a general truth from several particular truths. By induction we might infer that since the laws of §378 are true for every case we have examined, it will always be true.

380. **Mathematical Induction.**—Mathematical induction is the process of reasoning by which we establish a general truth or law by making use of a known fact and a conditional principle.

381. **Expansion** of $(a + b)^n$. In the foregoing we know that the laws hold for every case up to and including $n = 5$.

The condition is that we assume it true for any exponent **n,** and prove that it will hold true for the exponent **n + 1.**

1. **Assuming** $(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot2}a^{n-2}b^2 +$

$$\frac{n(n-1)(n-2)}{1\cdot2\cdot3}a^{n-3}b^3 + \ . \ . \ . \ +$$

$$\frac{n(n-1)(n-2)\ .\ .\ .\ [n-(k-2)]}{1\cdot2\cdot3\ .\ .\ .\ (k-1)}a^{n-(k-1)}b^{(k-1)} +$$

$$\frac{n(n-1)(n-2)\ .\ .\ .\ [n-(k-2)][n-(k-1)]}{1\cdot2\cdot3\ .\ .\ .\ (k-1)k}a^{n-k}b^k +$$

$$.\ .\ . + b^n.$$

Multiplying both sides of the equation by $(a + b)$ and collecting terms in the right hand member of the equation we have.

$$2.\ (a+b)^{n+1} = a^{n+1} + \frac{n+1}{1}a^nb + \frac{(n+1)(n)}{1\cdot 2}a^{n-1}b^2 + \frac{(n+1)(n)(n-1)}{1\cdot 2\cdot 3}a^{n-2}b^3 + \ldots + \frac{(n+1)(n)(n-1)\ \ldots\ [n+1-(k-2)]}{1\cdot 2\cdot 3\ \ldots\ k-1}a^{(n+1)-(k-1)}b^{k-1} \text{ etc.}$$

By comparing equations (1) and (2) it will be seen that every **n** in (1) is replaced by $(n + 1)$ in (2). This proves that if our expansion is true for the n^{th} power it is also true for the $(n + 1)^{\text{th}}$ power, thus making our expansion general.

Illustrations.—1. Expand $(2x - 3y)^4$.

Substituting $2x$ for **a**, $3y$ for **b**, and 4 for **n**, we have,

$$(2x-3y)^4 = (2x)^4 - 4(2x)^3(3y) + 6(2x)^2(3y)^2 - 4(2x)(3y)^3 + (3y)^4$$
$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4. \qquad \S 378.$$

2. Expand $\left(1 + \frac{1}{n}\right)^5$.

Solution.—$\left(1+\frac{1}{n}\right)^5 = 1^5 + 5(1)^4\left(\frac{1}{n}\right) + 10(1)^3\left(\frac{1}{n}\right)^2 + 10(1)^2\left(\frac{1}{n}\right)^3 + 5(1)\left(\frac{1}{n}\right)^4 + \left(\frac{1}{n}\right)^5.$

$$= 1 + \frac{5}{n} + \frac{10}{n^2} + \frac{10}{n^3} + \frac{5}{n^4} + \frac{1}{n^5}.$$

Note.—*In most cases the value of the coefficients of the expansion can be obtained mentally as the work proceeds.*

3. Expand $\left(1 + \frac{1}{x}\right)^x$ to 5 terms. Let x approach an infinite value and supply the result to 5 decimal places.

Solution.

$$\left(1+\frac{1}{x}\right)^x = 1^x + \frac{x}{1}\cdot 1^{x-1}\cdot\left(\frac{1}{x}\right) + \frac{x(x-1)}{1\cdot 2}\cdot 1^{x-2}\cdot\left(\frac{1}{x}\right)^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}1^{x-3}\cdot\left(\frac{1}{x}\right)^3 + \frac{x(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}1^{x-4}\left(\frac{1}{x}\right)^4 + \ldots$$

Simplifying we have, $1 + 1 + \frac{1}{2}\frac{x(x-1)}{x^2} + \frac{1}{2}\cdot\frac{1}{3}\frac{x(x-1)(x-2)}{x^3} + \ldots = 1 + 1 + \frac{1}{2}\left(1-\frac{1}{x}\right) + \frac{1}{6}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right) + \frac{1}{24}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right) + \ldots$

If we let x become infinite, $\frac{1}{x}, \frac{2}{x}$, etc., become 0.

Then $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} =$

$$\begin{array}{r} 1.00000 \\ 1.00000 \\ .50000 \\ .16667 \\ .04167 \\ \hline 2.70834 \end{array}$$

Note.—*In the above expansion the first and second terms are each 1. To get the third term, divide the second by 2. To get the fourth term, divide the third by 3. Continue this process until the quotient becomes 0 to as many decimal places as the work is carried out. Add the resulting terms.*

PROBLEMS

Expand the following:

1. $(m + n)^3$. 2. $(x - y)^3$. 3. $(a + 2b)^5$.

4. $(2a - x)^4$. 5. $\left(x + \frac{1}{x}\right)^4$. 6. $\left(m^2 - \frac{1}{2}\right)^3$.

7. $\left(1 + \frac{1}{x^2}\right)^2$. 8. $\left(1 - \frac{1}{b}\right)^4$. 9. $\left(x^2 - \frac{1}{4}\right)^2$.

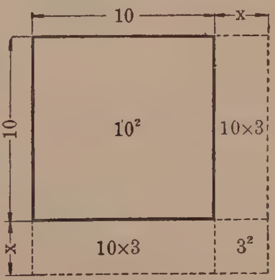
10. Expand $\left(1 + \frac{1}{n}\right)^n$ to 8 terms under the same conditions as in illustration 3. Simplify and carry the result to 5 decimals. Result 2.71828. This is an important number and is the base of the natural system of logarithms used in Calculus. It is designated by e.

382. **Square Root of Numbers.**—The square of a number contains twice as many figures as the number or twice as many less one. Thus, $5^2 = 25$; $25^2 = 625$; etc.

When we take the square root of a number we know how many figures will be in the result because we divide the number of which the root is required into periods of two figures each beginning at the right. If the number contains a decimal, we divide it into periods of two figures each both to the right and to the left of the decimal point.

Illustrations.—1. Find the square root of 169 or 100 + 69.

(*a*) Divide the number into periods, thus 1′69. The root will contain two figures.



(*b*) Build up a figure similar to the one in §221.

(*c*) We see that we have a square 10 by 10 or 100 square units.

(*d*) Subtracting 100 from 169 we have 69 square units remaining.

(*e*) We border the square (the dotted lines on the figure) and have 2 rectangles each 10 units long and a small square. The other dimension of the rectangle is found by dividing 20 into 69. It is contained three times, with a remainder. Therefore, the width of the rectangles is 3 units. The dimension of the square is also 3 units. Hence $2(10 \cdot 3) + 3^2 = 69$.

There is no remainder, therefore $10 + 3$ or 13 is the square root.

Assemble the above parts forming a square 13 by 13.

If there is a remainder again, we border the second square and continue in the same way.

From the above illustration we derive the following method:

$$\begin{array}{r|l|l}
 & 1'69 & 10 + 3 = 13 \\
10^2 = & 100 & \\
\text{Trial divisor } 2 \times 10 = 20 & 69 & \text{Remainder} \\
\text{Complete divisor } 20 + 3 = 23 & 69 & \\
 & \text{No remainder} &
\end{array}$$

In practice the zeros are omitted

$$\begin{array}{r|l|l}
 & 1'69 & 13 \\
 & 100 & \\
23 & 69 & \\
 & 69 &
\end{array}$$

2. Find the square root of 591361.

Solution.

$$\begin{array}{r|r|l}
 & 59'13'61 & 700 + 60 + 9 = 769 \\
 & & \text{Result} \\
(700)^2 = & 49\ 00\ 00 & \\
\text{Trial divisor } 2(700) = 1400 & 101361 & \text{Remainder} \\
\text{Complete divisor } 1400 + 60 = 1460 & 87600 & \\
 & 13761 & \text{Remainder} \\
\text{Trial divisor } 2(760) = 1520 & & \\
\text{Complete divisor } 1520 + 9 = 1529 & 13761 & \\
 & & \text{No remainder}
\end{array}$$

3. Find the square root of 146.41.

Solution.

$$\begin{array}{r|r|l}
 & 1'46.41 & 12.1 \\
 & 1 & \\
22 & 46 & \\
 & 44 & \\
 & 241 & \\
241 & 241 &
\end{array}$$

PROBLEMS

Find the square root of the following:

1. 625. 2. 729. 3. 3844. 4. 5776. 5. 9604.
6. 15625. 7. 10201. 8. .25. 9. 156.25. 10. 384.279.

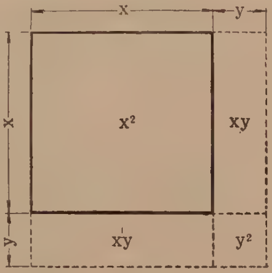
Find the square root of the following to 3 decimal places.

11. 2. 12. 3. 13. 5. 14. 2.5. 15. 3.5. 16. 12.

383. Square Root of Polynomials. Illustration.—Find the square root of $x^2 + 2xy + y^2$.

From our previous knowledge we know by inspection that the square root is $x + y$. To perform the operation step by step we recall the figure in §221.

We see from $x^2 + 2xy + y^2$ that it contains at least a square x by x. Subtracting x^2 from $x^2 + 2xy + y^2$ there remains $2xy + y^2$. We must now border our square by a strip wide enough to give an area $2xy + y^2$ or as near $2xy + y^2$ as possible. (The dotted line on the figure.)



There are two rectangles each x units long equivalent to one rectangle $2x$ units long. The width is not known. By dividing $2x$ into $2xy + y^2$ it is contained y times with a remainder. Therefore, y is the width. The area of the two rectangles is $2xy$. There will be a little square y by y needed to complete our bordered square with a dimension of $(x + y)$. $\therefore$ The border includes $2xy + y^2$. There is no remainder, hence $(x + y)$ is the required square root. If there is a remainder again, we border the second square and continue in the same way.

From the above illustration we derive a short method similar to that of §382.

Find the square root of $x^2 + 2xy + y^2$.

Solution.

$$\begin{array}{lr|ll}
 & & x^2 + 2xy + y^2 & \big|\; x + y \text{ Result} \\
 & (x)^2 = & x^2 & \\ \hline
\text{Trial divisor,} & 2x & \quad 2xy + y^2 & \text{Remainder} \\
\text{Complete divisor, } 2x + y & & \quad 2xy + y^2 & \\ \hline
 & & & \text{No remainder.}
\end{array}$$

PROBLEMS

Find the square root of the following:

1. $a^2 + 2ab + b^2$.
2. $9x^2 + 6x + 1$.
3. $25x^2 + 40xy + 16y^2$.
4. $4x^2 - 12xy + 9y^2$.
5. $x^4 + 2x^2y + y^2$.
6. $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$.

384. Solution of Equations by Square Root.

Illustrations.

1. Solve $x^2 = 9$.

Taking the square root of both members $x = \pm 3$.

2. Solve $4x^2 = 64$.

Solving as in (1), $2x = \pm 8$.

$\therefore x = \pm 4$.

3. Solve $b^2x^2 = c^2$.

Solving, $bx = \pm c$.

$\therefore x = \pm \frac{c}{b}$.

4. Solve $a^2 - x^2 = b^2$.

Solving, $x^2 = a^2 - b^2$.

$\therefore x = \pm\sqrt{a^2 - b^2}$.

Check each of the above.

PROBLEMS

Solve the following for x and check.

1. $x^2 = 16$.
2. $3x^2 = 75$.
3. $x^2 = 4b^2$.
4. $3x^2 = 2b$.
5. $2ax^2 = k^2$.
6. $m + x^2 = c$.
7. $2mx^2 - p = 0$.
8. $a^2x^2 = b^2$.
9. $5ax^2 - b = 0$.
10. $\frac{9}{2x} = 2x$.
11. $\frac{1}{2}ax^2 = m$.
12. $\frac{32}{a^2x} = 2x$.

13. What is the length of one side of a square field containing one acre? What is the perimeter? *Ans.* One side = 208.71 ft.

14. What is the length of one side of a square field containing 4 acres? What is the perimeter? *Ans.* One side = 417.42 feet.

15. Find the side of a square equivalent to a right triangle whose base is 20 feet and whose altitude is 22.5 feet. Which has the greater perimeter?

16. Find the side of a square equivalent to a rectangle whose base is 18 inches and whose altitude is 14 inches. Compare their perimeters.

Ans. One side of square = $6\sqrt{7}$.

17. How much more will it cost to enclose a rectangular field whose length is 25 rods and whose width is 16 rods, than a square containing the same area?

385. **Cube Root of Numbers.**—The cube root is shown because students invariably ask how to find the cube root of a number. The cube root of algebraic expressions is purposely omitted.

The cube of a number contains three times as many figures as the number, three times as many less one, or three times as many less two. Thus, $9^3 = 729$; $3^3 = 27$; $2^3 = 8$.

Therefore, when the cube root of a number is desired it is known how many figures there will be in the result because we divide the number into groups of three figures each beginning at the right.

Illustration.—Find the cube root of 1728.

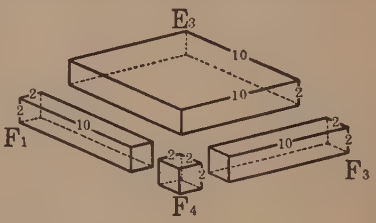
(*a*) Divide the number into periods, thus 1′728. The root will contain two figures.

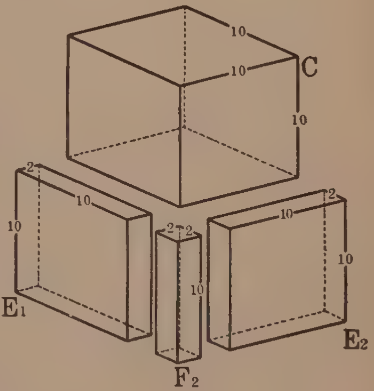
(*b*) Build up a cube to contain 1728 cubic units.

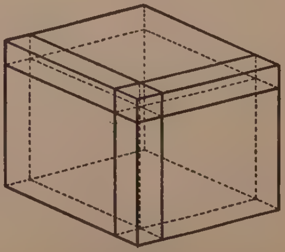
(*c*) We see that we have a cube 10 by 10 by 10 or 1000 cubic units (Fig. *C*).

(*d*) Subtracting 1000 from 1728 we have 728 cubic units left.

(*e*) Border the cube with three solids 10 by 10 by x (Figs. E_1, E_2, and E_3). To find what x is, we divide $3(10^2)$ or 300 units into 728 and find that it is con-







Completed Cube

tained twice with a remainder. Therefore 2 is the other dimension. Then we have 600 cubic units.

(*f*) To complete the cube we need four solids (Figs. F_1, F_2, F_3) with dimensions 10 by 2 by 2 units and (Fig. F_4) with dimensions 2 by 2 by 2 making $3(10 \times 2 \times 2) + 2^3$ or 128 cubic units.

(*g*) Subtracting we find no remainder. Therefore $(10 + 2)$ or 12 is the cube root.

In practice, proceed as follows. The zeros may be omitted.

$$\begin{array}{rr|l}
 & 1'728 & 10 + 2 = 12 \\
10^3 = & 1\,000 & \\
\hline
\text{Trial divisor } 3(10)^2 = 300 & 728 & \\
3(10 \times 2) = 60 & & \\
2 \times 2 = 4 & 728 & \\
\hline
\text{Complete divisor } = 364 & & \text{No remainder.}
\end{array}$$

PROBLEMS

Find the cube root of:

1. 2744.
2. 29791.
3. 300763.
4. 15 625.
5. 185193.
6. .068921.

CHAPTER XI

QUADRATIC EQUATIONS

386. **Equations Compared.**—In simple equations we had to deal with one unknown quantity. In simultaneous equations we had a system of equations involving two or more unknown quantities in two or more equations. However, in both simple and simultaneous equations the unknown quantities were of the first degree only. Now we are ready to advance to equations of the second degree.

387. **Quadratic Equations.**—A quadratic equation is one which involves the second degree of the unknown and no higher.

Quadratic equations are of two kinds, first those involving only the second degree of the unknown, and second those involving both the second and first degree of the unknown. Thus, $x^2 = 9$ is a quadratic equation involving only the second degree of x, the unknown.

$x^2 - x = 6$ is a quadratic equation involving both the second and the first degree of x.

388. **Solution of Quadratic Equations by Factoring.**—It was stated in §234 that if the product of any number of factors equals zero then one of the factors must be zero.

Illustrations.—Solve $x^2 = 9$.

1. **Solution.**

Transposing, $x^2 - 9 = 0$.
Factoring, $(x + 3)(x - 3) = 0$.
If $x + 3 = 0$, then $x = -3$.
If $x - 3 = 0$, then $x = 3$.

Check

When $x = -3$, $(-3)^2 = 9$.
When $x = 3$, $(3)^2 = 9$.

Therefore 3 and -3 are the two roots of the equation.

Note. $x^2 = 9$, *and all quadratic equations containing only the second degree of the unknown quantity can be solved by* §384.

2. Solve $x^2 - x - 6 = 0$.

Solution.

Factoring, $(x - 3)(x + 2) = 0$.
If $x - 3 = 0$, then $x = 3$.
If $x + 2 = 0$, then $x = -2$.

Check

When $x = 3$, $(3)^2 - 3 - 6 = 0$.
$9 - 3 - 6 = 0$.
When $x = -2$, $(-2)^2 + 2 - 6 = 0$.
$4 + 2 - 6 = 0$.

3. Solve $3x^2 - 8x - 3 = 0$.

Solution.

Factoring, $(3x + 1)(x - 3) = 0$.

Therefore, $x = -\frac{1}{3}$ or 3.

Check

When $x = -\frac{1}{3}$, $3(-\frac{1}{3})^2 - 8(-\frac{1}{3}) - 3 = 0$.
$\frac{1}{3} + \frac{8}{3} - \frac{9}{3} = 0$.
When $x = 3$, $3(3)^2 - 8(3) - 3 = 0$.
$27 - 24 - 3 = 0$.

4. Solve $\frac{y^2}{2} - \frac{y}{4} = \frac{3}{2}$.

Solution.

Clearing of fractions, $2y^2 - y = 6$.

Solving as before, $y = -\frac{3}{2}$ or 2.

Check both results in the original equation.

5. Solve $2x^2 - x = 0$.

Solution.

Factoring, $x(2x - 1) = 0$.

$\therefore$ $x = 0$ or $\frac{1}{2}$.

Note.—*Observe that when the constant term in a quadratic equation is zero, one of the roots is zero.*

389. **Summary of Solution by Factoring.**

1. Clear of fractions or remove parentheses when necessary.
2. Transpose all terms to one member.
3. Find the prime factors.
4. Equate each factor to zero and solve for the unknown.
5. Check by substituting in the original equation.

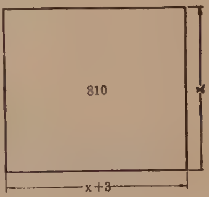
PROBLEMS

Solve the following quadratic equations by factoring. Check.

1. $m^2 + 3m - 10 = 0$.
2. $y^2 - 13y + 36 = 0$.
3. $p^2 - 7p = -6$.
4. $r^2 - 15 = 2r$.
5. $a^2 + 8a + 16 = 0$.
6. $x^2 - 7x = 0$.
7. $4c^2 - 3c = 0$.
8. $x^2 - 2x = 48$.
9. $d^2 - 5d + 4 = 0$.
10. $15x^2 - 2 = -7x$.
11. $49x^2 - 64 = 0$.
12. $5x^2 = 80$.
13. $12c^2 - 17c = -6$.
14. $d^2 - 6d + 9 = 0$.
15. $6p^2 + p - 35 = 0$.
16. $56y^2 - 6y - 2 = 0$.
17. $x^2 + \frac{3}{2}x - 1 = 0$.
18. $2x^2 - \frac{5}{3}x - 2 = 0$.
19. $x^2 - x = -\frac{2}{9}$.
20. $15x^2 + 8x = 63$.

21. Lay out a floor to contain 810 square feet so that the length is 3 feet greater than the width.

Note.—*Draw a figure whenever possible to help analyze the problem.*



Solution.

Let x = the width of floor.

Then $x + 3$ = the length of floor.

Hence, $x(x + 3)$ = the area or lw.

Then $x(x + 3) = 810$ or $x^2 + 3x - 810 = 0$.

Factoring, $(x + 30)(x - 27) = 0$.

$\therefore x = -30$ or 27.

Since a floor minus 30 feet long is impossible, we take only the positive value 27, which satisfies the conditions of the problem.

Check. $30 \times 27 = 810$ and 30 feet, the length, is 3 more than 27 feet, the width.

22. Twice the square of a given number is 32. Find the number.

23. Divide 20 into two parts so that the sum of the squares will equal 208.

24. A triangle contains 9 sq. in. Its altitude is 3 inches longer than the base. Find its dimensions.

25. A plot of ground is 28′ × 20′. It is desired to reduce the size to 384 sq. ft. by shortening each dimension by the same amount. Determine this amount.

26. A factory floor 80′ × 60′ is partly taken up by an aisle of uniform width extending along the four sides. The area of the aisle is one half the entire floor. Find the width of the aisle.

27. Find two consecutive integers whose product is 420.

28. Find two consecutive even integers whose product is 168.

29. Find two numbers whose difference is 3 and whose product is 108.

30. $\frac{3}{x^2} - \frac{1}{x^2} = 8.$

31. $\frac{7}{2} - \frac{5}{m} = \frac{4}{m^2}.$

32. $6x + 11 - \frac{10}{x} = 0.$

33. $\frac{2}{a^2} = \frac{3}{a} - 1.$

34. $\frac{a^2 + a + 1}{a - 1} - \frac{a^2 - a + 1}{a + 1} = 6.$

35. $\frac{2m - 3}{m + 1} = \frac{4 + 2m}{3m}.$

36. $\frac{2x^2 + 2}{2} - \frac{2x + 4}{3} = \frac{7}{3}.$

37. $\frac{5m - 2}{3m + 5} = \frac{8(m - 1)}{2m + 7}.$

38. $\frac{2a + 3}{a - 1} - \frac{a + 1}{2a - 3} = 4.$

39. $x^2 - 3cx + 2c^2 = 0.$

Solution. $(x - c)(x - 2c) = 0.$

$\therefore$ $x = c$ and $2c$.

40. $a^2x^2 - b^2 = 0.$

41. $x^2 - bx = 12b^2.$

42. $2m^2 - 3mx + x^2 = 0.$

43. $2x^2 - 3bx + b^2 = 0.$

44. $6x^2 - 5bx + b^2 = 0.$

390. Completing the Square.—It was learned in chapter V that $x^2 + 8x + 16$ is the square of $x + 4$ and $x^2 - 8x + 16$ is the square of $x - 4$. Therefore, to make squares of $x^2 + 8x$ and $x^2 - 8x$, it is necessary to add 4^2 to each expression.

Thus, to complete the square of a binomial, when the coefficient of x^2 is 1, add the square of one-half the coefficient of x, where x is the unknown.

PROBLEMS

Complete the square of the following:

1. $x^2 + 2x.$
2. $a^2 - 6a.$
3. $x^2 + 2cx.$
4. $a^2 - ab.$
5. $x^2 + 16x.$
6. $x^2 - 9x.$
7. $x^2 + 4x.$
8. $x^2 - \frac{3}{2}x.$
9. $x^2 + \frac{3}{5}x.$
10. $x^2 - \frac{5}{2}x.$
11. $x^2 + \frac{1}{2}x.$
12. $x^2 + 9x.$
13. $x^2 - \frac{1}{3}x.$
14. $y^2 - 3y.$
15. $x^2 - 7x.$
16. $x^2 - \frac{c}{a}x.$
17. $x^2 - 2(a - b)x.$
18. $x^2 + (a - b)x.$
19. $x^2 - \frac{2}{c}x.$
20. $x^2 - (a - b)x.$
21. $x^2 - x.$

391. Solution of Quadratic Equations by Completing the Square.—We shall now apply this method to the solution of quadratic equations which are not readily factored.

Illustrations.

1. Solve $x^2 + 6x + 8 = 0.$

Solution.

Transposing 8, $x^2 + 6x = -8.$

Squaring $\frac{1}{2}$ the coefficient of x, and adding to both members, $x^2 + 6x + 9 = 9 - 8.$

$$(x + 3)^2 = 1.$$

Taking the square root of both members, $x + 3 = \pm 1.$

Solving for x, $x = -3 + 1$ or $-3 - 1.$

Simplifying, $x = -2$ or $-4.$

Check these two roots.

2. Solve $3x^2 - 2x - 4 = 0.$

Solution.

Dividing by 3 for coefficient of x^2 must be 1, $x^2 - \frac{2}{3}x - \frac{4}{3} = 0.$

Transposing, $x^2 - \frac{2}{3}x = \frac{4}{3}.$

Squaring $\frac{1}{2}$ the coefficient of x and adding to both members, $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}.$

$$(x - \tfrac{1}{3})^2 = \tfrac{4}{3} + \tfrac{1}{9} = \tfrac{13}{9}.$$

Taking the square root of both members $x - \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$ or $\pm\frac{1}{3}\sqrt{13}.$

$$\therefore \quad x = \tfrac{1}{3} \pm \tfrac{1}{3}\sqrt{13}.$$

$$\therefore \quad x = \frac{1 \pm \sqrt{13}}{3}.$$

Check these roots.

392. **Summary of Solution by Completing the Square.**

1. Make the coefficient of the square of the unknown quantity unity.

2. Transpose the constant terms.

3. Add to each member of the equation the square of one-half the coefficient of the unknown of the first degree.

4. Take the square root of both sides of the equation.

5. Solve the resulting simple equation.

PROBLEMS

Solve the following quadratic equations by "Completing the Square."

1. $x^2 - 2x + 1 = 0.$ 2. $x^2 - 3x + 2 = 0.$ 3. $a^2 - 2a - 4 = 0.$
4. $m^2 - m = 5.$ 5. $m^2 - m = 6.$ 6. $m^2 + m - 30 = 0.$
7. $y^2 - 10y - 20 = 0.$ 8. $x^2 + 10x + 24 = 0.$ 9. $x^2 - 9x = 11.$
10. $x^2 - \frac{2}{3}x = 1.$ 11. $m^2 + 10m = -15.$ 12. $r^2 + 3r + 1 = 0.$
13. $a^2 + 5a = 1.$ 14. $5x + 6 = x^2.$ 15. $y^2 - \frac{1}{3}y = 2.$
16. $b^2 - \frac{3}{5}b = \frac{1}{5}.$ 17. $z^2 + \frac{3}{7}z - 2 = 0.$ 18. $m^2 + \frac{1}{5}m = 10.$
19. $x^2 + \frac{1}{5}x - 9 = 0.$ 20. $r^2 + \frac{3}{8}r = \frac{1}{8}.$

Divide problems 21–26 by the coefficient of x^2 and solve.

21. $2x^2 - 5x - 3 = 0.$ 22. $4x^2 + 4x = 3.$ 23. $3x^2 - x - 1 = 0.$
24. $3x^2 - 7x + 2 = 0.$ 25. $5x^2 + 13x + 10 = 0.$ 26. $2x^2 - 5x + 7 = 0.$

Clear of fractions and solve.

27. $\frac{3}{a^2} + \frac{1}{a} = \frac{3}{2}.$ 28. $\frac{3x}{2} - \frac{2}{x} = 3.$

29. $\frac{5}{r^2} + \frac{3}{r} = 6.$

30. $\frac{2x}{x+2} = \frac{5}{x-1}.$

31. $\frac{3x}{x-7} + \frac{2}{x-4} = 0.$

32. $\frac{2}{x+2} - \frac{3}{x+1} = \frac{2}{3}.$

33. $\frac{2x-1}{3x+2} - \frac{2x}{x-1} = 1.$

34. $\frac{1}{3a+1} - 2 = \frac{2}{3a+2}.$

35. $\frac{2}{x^2-9} - \frac{3x}{x+3} = \frac{1}{x-3}.$

36. $\frac{4m}{6m-3} + \frac{2m-7}{2m+3} = 1.$

37. $\frac{3}{x^2+3x+2} + \frac{4}{x^2+4x+3} = \frac{5}{x^2+6x+8}.$

393. **Derivation of a General Formula for Solving any Quadratic Equation.**—The general form for all quadratic equations in one unknown is $ax^2 + bx + c = 0$, in which **x** is the unknown, **a** is the coefficient of x^2, **b** is the coefficient of x, and **c** is the constant or absolute term.

The **formula** is derived by solving the above quadratic equation by completing the square as follows:

Solve $\quad ax^2 + bx + c = 0.$

Transposing **c**, $\quad ax^2 + bx = -c.$

Dividing by **a**, $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Adding the square of ½ the coefficient of x to both members, $\quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$

Then $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$ $\quad$ Why?

Taking the square root of both members,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Solving for x, $\quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$

Simplifying, $\quad \mathbf{x} = \frac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 4\mathbf{ac}}}{2\mathbf{a}}$

Therefore the two values of x are, $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}.$

394. **Solution of Quadratic Equations by the Formula.**

Illustrations.

1. Solve $3x^2 + 2x - 5 = 0.$

Solution.—Comparing this equation with $ax^2 + bx + c = 0$, we have $a = 3$, $b = 2$, and $c = -5$.

Substituting in the formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we have,

$$x = \frac{-2 \pm \sqrt{4 - 4(3)(-5)}}{2 \cdot 3} = \frac{-2 \pm 8}{6}.$$

$$\therefore x = 1 \text{ or } -\tfrac{5}{3}.$$

$\pm\sqrt{b^2-4ac}$

Therefore the roots of the given equation are 1 and $-\frac{5}{3}$.

Check.

When $x = 1$,
$3(1)^2 + 2(1) - 5 = 0.$
$3 + 2 - 5 = 0.$

When $x = -\frac{5}{3}$
$3(-\frac{5}{3})^2 + 2(-\frac{5}{3}) - 5 = 0.$
$\frac{25}{3} - \frac{10}{3} - \frac{15}{3} = 0.$

PROBLEMS

Solve the following by the formula.

1. $x^2 + 2x - 3 = 0.$
2. $x^2 + 2x + 1 = 0.$
3. $x^2 + 3x - 4 = 0.$
4. $x^2 + 2x - 1 = 0.$
5. $x^2 + 2x - 8 = 0.$
6. $2x^2 + 4x - 6 = 0.$
7. $3m^2 - 7m + 3 = 0.$
8. $y^2 - 72 = 6y.$
9. $2m^2 - 8m - 3 = 0.$
10. $y^2 + y - 1 = 0.$
11. $4m^2 - 7m + 3 = 0.$
12. $12x^2 + x = 6.$
13. $6x^2 + x - 12 = 0.$
14. $\frac{3}{x - 3} - 2x = 0.$
15. $x^2 - 6x = -9.$
16. $\frac{x^2}{2} - \frac{x}{3} = 5.$
17. $ax^2 - 2bx + k = 0.$
18. $bx^2 + 2mx + q = 0.$
19. $cx^2 + 3rx + b = 0.$
20. $ax^2 + bx + c = 0.$
21. $3y^2 + 5y - 7 = 0.$
22. $\frac{4}{3 - y} - \frac{3}{y} = 2.$
23. $\frac{3}{x} - x = \frac{1}{4}.$
24. $\frac{7x}{x + 1} + \frac{3x}{4} = 5.$
25. $y^2 - 16 = 0.$
26. $x^2 - 2 = 0.$
27. $x^2 - 25 = 0.$
28. $x^2 + x\sqrt{5} = 0.$
29. $x^2 - 5x = 0.$
30. $a^2x^2 - d^2 = 0.$
31. Twice the square of a certain number diminished by 8 equals the product of the number and 15. Find the number.
32. A rectangle has an area of 60 sq. ft. Its length is 4 ft. greater than its width. Find its dimensions.
33. The sum of the squares of two consecutive integers is 685. Find the integers.
34. The sum of the squares of two consecutive odd integers is 202. Find the integers.
35. The sum of a certain number and its reciprocal is $\frac{17}{4}$. Find the number.
36. The area of a triangle is 150 sq. in. The altitude is 5 more than $\frac{1}{2}$ the base. Find the base and altitude.
37. A rectangle and a square have equal areas. The base of the rectangle is 5 ft. longer than the altitude. The side of the square is $\sqrt{150}$. Find the dimensions of the rectangle.
38. The sum of the base and altitude of a rectangle is 28. The area is 195 sq. ft. Find its dimensions.
39. A rectangular plot of ground is 80 rods long and 50 rods wide. A strip around it of uniform width is staked off reducing the area by $\frac{3}{10}$. Find the width of this strip.
40. The force in dynes exerted between two charged spheres varies directly as the product of their charges and inversely as the square of the distance between them in air.

Formula

$$F = \frac{QQ'}{kd^2}.$$

F = force in dynes of repulsion or attraction.
Q = charge in units of one sphere.
Q' = charge in units of other sphere.

$k = 1$, the constant for air.
$d =$ distance in centimeters.
1 inch = 2.54 centimeters.

(*a*) Find the distance in inches between two insulated brass spheres if the charge on one is 100 units, and on the other 50 units, when placed in air if the force of repulsion is 25 dynes.

(*b*) Make both charges the same value, for example have both equal to 100.

395. Approximate Values of Roots.—For practical purposes it is necessary to express roots in rational rather than in irrational form, that is, in approximate or exact numerical values.

Thus, the roots of the equation, $x^2 - 4x + 1 = 0$ are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

But $2 + \sqrt{3} = 2.000 + 1.732 = 3.732$.
and $2 - \sqrt{3} = 2.000 - 1.732 = 0.268$.

Here the values 3.732 and 0.268 are numerical values and while not absolutely exact are so nearly exact that engineers are able to apply these values in practical work.

PROBLEMS

Find the value of the roots of the following equations to three decimal places.

1. $x^2 - 4x + 1 = 0$. 2. $x^2 - 8x + 7 = 0$. 3. $2x^2 - 7x - 2 = 0$.
4. $x^2 - 8x + 4 = 0$. 5. $5x^2 - 6x = 1$. 6. $3x^2 - 2x - 3 = 0$.
7. $2x^2 - 11x + 5 = 0$. 8. $3x^2 + 4x - 9 = 0$. 9. $10x^2 + 3x = \frac{17}{20}$.
10. $5x^2 - 3x = \frac{1}{4}$.

396. Equations in the Quadratic Form.—In equations of this type the value of the expression containing the unknown is found first. The resulting equations are then solved for the unknown.

Illustration. $(x + 4)^2 + 4(x + 4) = 60$ is a quadratic in $(x + 4)$.

Solution by the formula.

$$(x + 4)^2 + 4(x + 4) - 60 = 0.$$

$$\therefore (x+4) = \frac{-4 \pm \sqrt{16 + 240}}{2}.$$

$$\therefore x + 4 = 6 \text{ or } -10.$$

$$\therefore x = 2 \text{ or } -14.$$

Check both roots.

PROBLEMS

Solve the following and check.

1. $(x + 3)^2 + (x + 3) = 30$. 2. $(x + 2)^2 + 2(x + 2) = 24$.
3. $(x - 1)^2 - 3(x - 1) = 0$. 4. $(x^2 - x)^2 - 2(x^2 - x) = 24$.
5. $(x^2 + 3)^2 - 2(x^2 + 3) = 35$. 6. $(x + 2)^2 - 6(x + 2) = 7$.
7. $(x^2 + 4)^2 - 2(x^2 + 4) = 143$. 8. $(x^2 - 1)^2 - 3(x^2 - 1) - 40 = 0$.
9. $x + 5 + \sqrt{x + 5} = 20$. 10. $x + 3 - \sqrt{x + 3} = 6$.

397. Irrational Equations.—An irrational equation is one in which the unknown appears under the radical sign.

Illustrations. Check

1. Solve $\sqrt{x} - 2 = 5$. $\sqrt{49} - 2 = 5$.
Solution.—Transposing, $\sqrt{x} = 7$. $7 - 2 = 5$.
Squaring both sides $x = 49$. $5 \equiv 5$.

2. Solve $4 - \sqrt{3x - 5} = 2$.

Solution.—Transposing, $2 = \sqrt{3x - 5}$.

Squaring, $4 = 3x - 5$.

$\therefore$ $9 = 3x$.

$3 = x$

Check

$4 - \sqrt{9 - 5} = 2$.

$4 - 2 = 2$.

$2 \equiv 2$.

3. Solve $x + \sqrt{9x + 4} = 12$.

Solution.—Transposing, $\sqrt{9x + 4} = 12 - x$.

Squaring, $9x + 4 = 144 - 24x + x^2$.

Transposing, $x^2 - 33x + 140 = 0$.

Factoring, $(x - 5)(x - 28) = 0$.

$x = 5$ or 28.

Check

When $x = 5$, $5 + \sqrt{45 + 4} = 12$.

$5 + 7 = 12$.

$12 \equiv 12$.

When $x = 28$, $28 + \sqrt{252 + 4} = 12$.

$28 + 16 \neq 12$.

In checking we find that the root 5 satisfies the equation, but the root 28 does not check when using the principal root of 256. Therefore we accept root 5 as true. 28 is called an **extraneous root**. See §398.

4. Solve $\sqrt{4x + 1} - \sqrt{3x - 2} = 1$.

Solution.—Transposing, $\sqrt{4x + 1} = 1 + \sqrt{3x - 2}$.

Squaring, $4x + 1 = 1 + 2\sqrt{3x - 2} + 3x - 2$.

Transposing, $x + 2 = 2\sqrt{3x - 2}$.

Squaring, $x^2 + 4x + 4 = 12x - 8$.

Transposing, $x^2 - 8x + 12 = 0$.

Factoring, $(x - 2)(x - 6) = 0$.

$\therefore$ $x = 2$ or 6.

Check

When $x = 2$, $\sqrt{8 + 1} - \sqrt{6 - 2} = 1$.

$3 - 2 = 1$.

$1 = 1$.

When $x = 6$, $\sqrt{24 + 1} - \sqrt{18 - 2} = 1$.

$5 - 4 = 1$.

$1 = 1$.

Both roots satisfy the equation.

398. **Extraneous Roots.**—An extraneous root is a root which will satisfy the given equation only when the sign opposite to the sign of the principal root is used in the check. Such a root is often introduced when both members of an equation are squared. Thus in the solution of

illustration (3) §397, the roots found are 5 and 28. The root 5 satisfies the equation $x + \sqrt{9x + 4} = 12$, while 28 does not satisfy it, unless -16 is used for the square root of 256. The root 28 is therefore an extraneous root.

A study of the following problem will show how an extraneous root may enter into the problem.

Let $x = 2$.
Squaring, $x^2 = 4$.
$\therefore$ $x = \pm 2$.

Note that $+2$ satisfies the equation $x = 2$, while -2 does not. Therefore -2 in an extraneous root.

PROBLEMS

Solve the following irrational equations and check.

1. $\sqrt{x - 1} - 2 = 0$.
2. $\frac{1}{\sqrt{x}} = 1$.
3. $\frac{1}{\sqrt{x + 1}} = 3$.
4. $\sqrt{3x - 2} + 2 = 4$.
5. $\sqrt[3]{x + 5} + 1 = 3$.
6. $\sqrt{3x + 4} - 5 = -1$.
7. $x + \sqrt{x} = 6$.
8. $x + \sqrt{2x - 1} = 8$.
9. $\frac{1}{\sqrt{3x - 2}} - 1 = -\frac{3}{4}$.
10. $\sqrt[3]{x^2 - 8} = 2$.
11. $3\sqrt{x} - \sqrt{x + 17} = 15$.
12. $\sqrt{x - 7} = 2\sqrt{x - 5}$.
13. $\sqrt{x - 11} + \sqrt{x - 9} = 2$.
14. $\sqrt{8x + 1} = 1 + \sqrt{6x - 2}$.

THEORY OF QUADRATIC EQUATIONS

399. Nature of the Roots of a Quadratic Equation.—Consider the nature of the roots of the following equations:

Equation	Roots	Nature
1. $x^2 = 9$.	$3, -3$.	Real, Rational, Unequal.
2. $x^2 - 6x + 9 = 0$.	$3, \quad 3$.	Real, Rational, Equal.
3. $5x^2 - 7x + 1 = 0$.	$\frac{7 \pm \sqrt{29}}{10}$.	Real, Irrational, Unequal.
4. $2x^2 + 2x + 5 = 0$.	$\frac{-2 \pm \sqrt{-36}}{4} =$ $\frac{-1 \pm 3\sqrt{-1}}{2}$.	Imaginary, Unequal.

400. Discriminant.—The discriminant of the quadratic formula is $b^2 - 4ac$. It enables us to determine the nature of the roots. Test each one of the above equations for the nature of its roots by substituting the numerical values for a, b, and c.

Equation	Discriminant	Roots
1. In $x^2 + 0 - 9 = 0$.	$0 - 4(-9) = 36$.	$3, -3$.
2. In $x^2 - 6x + 9 = 0$.	$36 - 4(9) = 0$.	$3, \quad 3$.
3. In $5x^2 - 7x + 1 = 0$.	$49 - 4(5)(1) = 29$.	$\frac{7 \pm \sqrt{29}}{10}$.
4. In $2x^2 + 2x + 5 = 0$.	$4 - 4(2)(5) = -36$.	$\frac{-1 \pm 3\sqrt{-1}}{2}$.

In (1), $b^2 - 4ac$ is a square. The roots are real, rational, and unequal.

In (2), $b^2 - 4ac$ equals zero. The roots are real, rational, and equal.

In (3), $b^2 - 4ac$ is not a perfect square and is greater than zero. The roots are real, irrational, and unequal.

In (4), $b^2 - 4ac$ is less than zero. The roots are imaginary and unequal.

PROBLEMS

Find by inspection the nature of the roots of the following equations.

1. $x^2 - 10x + 25 = 0.$
2. $x^2 - 16 = 0.$
3. $3x^2 + x - 2 = 0.$
4. $x^2 - x - 6 = 0.$
5. $2x^2 + x + 5 = 0.$
6. $4x^2 - 9 = 0.$
7. $15x^2 + 29x - 14 = 0.$
8. $x^2 + x + 7 = 0.$
9. $9x^2 - 12x + 4 = 0.$
10. $x^2 + 8x + 16 = 0.$

401. The Sum and the Product of the Roots of a Quadratic Equation.

Given the equation, $ax^2 + bx + c = 0.$ (1)

Dividing by a, $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \qquad (2)$$

Substituting p for $\frac{b}{a}$ and q for $\frac{c}{a}$ in (2) we have,

$$x^2 + px + q = 0. \qquad (3)$$

Solving (3) by the formula, $$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Representing the roots by x_1 and x_2 we have,

$$x_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \text{ and } x_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Adding, $$x_1 + x_2 = \frac{-2p}{2} = -p.$$

Multiplying, $$x_1 x_2 = \frac{p^2 - (p^2 - 4q)}{4} = \frac{4q}{4} = q.$$

Hence, the sum of the roots of the equation $x^2 + px + q = 0$ is the coefficient of x with the sign changed, and the product of the roots is the constant term.

Illustration.—Find the sum and the product of the roots of the equation, $5x^2 + 9x + 4 = 0.$

Solution.—Dividing by 5, $x^2 + \frac{9}{5}x + \frac{4}{5} = 0.$

It is seen by inspection that the sum of the roots is $-\frac{9}{5}$, or the coefficient of x with the sign changed.

The product of the roots is $\frac{4}{5}$, or the constant term.

To check this we find the roots.

$$5x^2 + 9x + 4 = 0.$$
$$(5x + 4)(x + 1) = 0.$$
$$\therefore x = -\tfrac{4}{5} \text{ and } -1.$$
$$\text{Sum} = (-\tfrac{4}{5}) + (-1) = -\tfrac{9}{5}.$$
$$\text{Product} = (-\tfrac{4}{5})(-1) = \tfrac{4}{5}.$$

PROBLEMS

Find the sum and the product of the roots of the following quadratic equations. Check 1 to 6 by actual solution.

1. $m^2 + 3m - 10 = 0$. 2. $y^2 - 13y + 36 = 0$.
3. $p^2 - 7p = -6$. 4. $r^2 - 15 = 2r$.
5. $5x^2 = 80$. 6. $12c^2 - 17c = -6$.
7. $d^2 - 6d + 9 = 0$. 8. $6p^2 + p - 35 = 0$.
9. $a^2 + 8a + 16 = 0$. 10. $x^2 - 2x = 48$.
11. $15x^2 - 2 = -7x$. 12. $4c^2 - 3c = 0$.
13. $3a^2 - 6a = 0$. 14. $x^2 - x = -\frac{2}{9}$.

402. Formation of Quadratic Equations when the Roots are Known. There are two simple methods.

Illustrations.

(*a*) By the converse of §401.

The sum of the roots of a quadratic equation is $-\frac{1}{4}$ and the product is $-\frac{3}{4}$. Form the equation.

From the equation $x^2 + px + q = 0, p = \frac{1}{4}$ and $q = -\frac{3}{4}$.

Therefore $x^2 + \frac{1}{4}x - \frac{3}{4} = 0$.

Clearing of fractions, $4x^2 + x - 3 = 0$ is the required quadratic.

(*b*) By the converse of §388.

Solving the equation, $x^2 - x - 6 = 0$.

$$(x - 3)(x + 2) = 0.$$

$$\therefore x = 3 \text{ or } -2.$$

Observe that the signs of -3 *and* $+2$ *in the factors of the equation are both changed in the roots* $+3$ *and* -2.

Therefore, to form the equation, incorporate the roots with the signs changed in two binomial factors thus, $(x - 3)$ *and* $(x + 2)$. *Find their product and set it equal to zero. Hence* $x^2 - x - 6 = 0$ *is the required equation.*

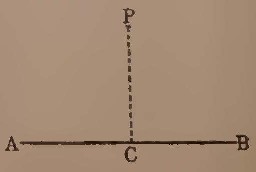
This method may be used to form equations of any degree.

PROBLEMS

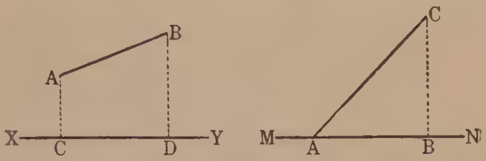
Form the equations whose roots are:

1. $3, -2$. 2. $4, -7$. 3. $\frac{1}{3}, -\frac{3}{4}$. 4. $\frac{a}{2}, \frac{a}{3}$.
5. $-4, -5$. 6. $6, 8$. 7. $2a, -3a$. 8. $2, 4, -6$
9. $2, 3$. 10. $-5, 6$. 11. $-7, -8$. 12. $11, -4$.
13. $6, -8$. 14. $1, -\frac{2}{3}$. 15. $\frac{1}{2}, \frac{3}{4}$. 16. $4b, -7b$.
17. $-3c, -2c$. 18. $2r - s, 2r + s$. 19. $2 + \sqrt{5}, 2 - \sqrt{5}$.
20. $2 + 3\sqrt{2}, 2 - 3\sqrt{2}$.

403. Projection of a Point upon a Line.—The projection of a point upon a straight line is the foot of the perpendicular from that point to the line. Thus, C is the projection of the point P upon the line AB.



404. Projection of a Line upon a Line.—The projection of a line upon a second line is the segment of the second line included between the perpendiculars drawn from the extremities of the first line to the second line.



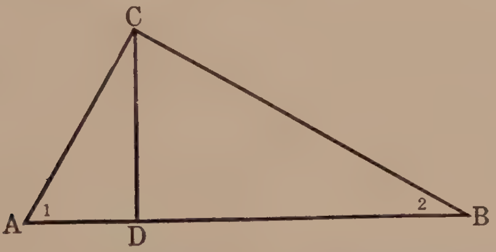
Thus, CD is the projection of AB upon XY and AB is the projection of AC on MN.

405. Theorem.—If, in a right triangle, an altitude is drawn to the hypotenuse:

1. The two triangles thus formed are similar to the given triangle, and to each other.

2. The altitude is a mean proportional between the segments of the hypotenuse.

3. Each leg is a mean proportional between the whole hypotenuse and the segment of the hypotenuse adjacent to that leg.



Given the rt. $\triangle\ ABC$ in which $CD \perp$ to AB, the hypotenuse.

To prove that $\triangle$s ABC, ADC, and DBC are similar.

Proof.—(1) In the rt. $\triangle$s ADC and ABC, $\angle\ 1$ is common.

$\therefore$ The $\triangle$s ADC and ABC are similar. §311.

In the rt. $\triangle$s ABC and DBC, $\angle\ 2$ is common.

What then is true of these $\triangle$s? Why?

Then the rt. $\triangle$s ADC and DBC are similar.

(Two $\triangle$s similar to a third $\triangle$ are similar to each other.)

(2) The rt. $\triangle$s ADC and DBC are similar. (1)

$\therefore AD:DC = DC:DB$. §307.

(3) The rt. $\triangle$s ABC and ADC are similar. (1)

$\therefore AB:AC = AC:AD$. Why?

The rt. $\triangle$s ABC and DBC are similar. (1)

$\therefore AB:BC = BC:DB$. Why?

Q.E.D.

406. Corollary 1.—If two chords are drawn from the same point on the circumference of a circle to the extremities of a diameter, either chord is a mean proportional between the diameter and the projection of the chord upon the diameter.

407. **Corollary 2.**—The perpendicular drawn from any point on a circumference to the diameter is a mean proportional between the segments of the diameter.

408. **Corollary 3.**—In any right triangle the product of the legs is equal to the product of the hypotenuse and the altitude on the hypotenuse.

In the Fig. of §405, △s ADC and ABC are similar.

Then $\frac{AC}{AB} = \frac{CD}{BC}$. $\therefore AC \times BC = CD \times AB.$

409. **Corollary 4.**—The squares of the legs of a right triangle are proportional to their projections upon the hypotenuse.

In the Fig. of §405, $\overline{CA}^2 = AB \times AD$, and $\overline{CB}^2 = AB \times BD$.

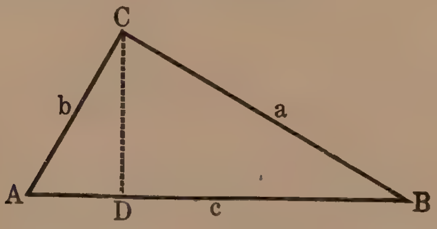
$$\therefore \frac{\overline{CA}^2}{\overline{CB}^2} = \frac{\cancel{AB} \times AD}{\cancel{AB} \times BD} = \frac{AD}{BD}.$$

410. **Corollary 5.**—The altitude upon the hypotenuse is equal to the product of the legs divided by the hypotenuse.

In the Fig. of §405, △s ABC and ADC are similar.

Then $\frac{AB}{AC} = \frac{BC}{CD}$. $\therefore AB \times CD = AC \times BC$ or $CD = \frac{AC \times BC}{AB}$.

411. **Theorem.**—In any right triangle the sum of the squares of the legs is equal to the square of the hypotenuse.



Given AC and CB the legs and AB the hypotenuse of the rt. △ ABC.

To prove that $\overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2$.

Proof.—Draw $CD \perp AB$.

Then $\overline{AC}^2 = AB \times AD$. §405–3.

And $\overline{CB}^2 = AB \times BD$. Why?

Adding $\overline{AC}^2 + \overline{CB}^2 = AB(AD + BD)$

or $\overline{AC}^2 + \overline{CB}^2 = AB \times AB = \overline{AB}^2$. Q.E.D.

412. **Corollary.**—In any right triangle the square of either leg is equal to the square of the hypotenuse minus the square of the other leg.

Denoting the sides of the △ ABC by a, b, and c as shown we have, $a^2 + b^2 = c^2 \therefore a^2 = c^2 - b^2$ and $a = \sqrt{c^2 - b^2}$.

Also $b^2 = c^2 - a^2$ and $b = \sqrt{c^2 - a^2}$.

PROBLEMS

1. Find the hypotenuse of a right triangle, if the legs are 15 and 20.

2. A carpenter wishes to square the frame work of sills to be laid on the foundation walls. He takes three floor boards 12, 16 and 20 feet respectively and nails them together forming a triangle. He makes the angle of his frame work equal

to the angle formed by the 12 and 16 foot sides of his triangle. Why is he sure to have a square corner?

3. The hypotenuse of a right triangle is 12 and one leg is 9. Find the other leg.

4. Find the diagonal of a square whose side is 1 foot.

What is the ratio of this diagonal to a side of the square?

Can the value $\sqrt{2}$ be exactly determined?

Is this an example of an incommensurable ratio?

5. A rectangle is 13 by 7. Find its diagonal.

6. The sides of a right triangle are 6, 8, and 10. Find the altitude on the hypotenuse.

7. The legs of a right triangle are 8 and 15. Find the altitude on the hypotenuse.

8. The legs of a right triangle are 8 and 12. Find the hypotenuse; the altitude upon the hypotenuse; and the projection of each leg upon the hypotenuse.

9. The leg of an isosceles triangle is 15 and the base is 10. Find the altitude.

10. The base of an isosceles triangle is 10 and the altitude is 5. Find the equal sides.

11. Find the altitude of an equilateral triangle whose side is 8.

12. Find the altitude **h** of an equilateral triangle whose side is x. Using the formula here derived, find the altitude of an equilateral triangle whose side is 20.

13. The altitude of an equilateral triangle is 12 inches. Find a side.

14. If the altitude of an equilateral triangle is h, find a side.

15. How long is a ladder which reaches a window 30 feet high on the side of a building, if the foot of the ladder is 9 feet from the wall?

16. Two buildings on opposite sides of 30-foot street are 24 and 36 feet high respectively. Find the air-line distance between their tops.

17. One leg of a right triangle is **m** and the hypotenuse is **h**. Find the length of the other leg.

18. The perimeter of a rectangle is **p** and the diagonal is **d**. Find the length and width of the rectangle. *Ans.* length $= \dfrac{p \pm \sqrt{8d^2 - p^2}}{4}$.

$$\text{width} = \frac{p}{2} - \text{length} = \frac{p \mp \sqrt{8d^2 - p^2}}{4}.$$

413. **Theorem.**—In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of those sides and the projection of the other upon that side.

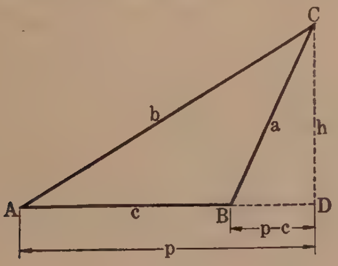


FIG. 1.

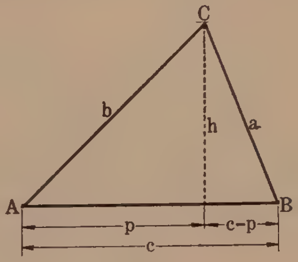


FIG. 2.

Given the acute $\angle A$ in the $\triangle ABC$ and AD (or p) the projection of AC on AB in either $\triangle$.

To prove that $a^2 = b^2 + c^2 - 2cp$.

Proof.—In Fig. 1, $a^2 = h^2 + (p - c)^2$. (1) Why?

Also $h^2 = b^2 - p^2$. (2) Why?

Substituting the value of h^2 from (2) in (1) and simplifying, $a^2 = b^2 + c^2 - 2cp$.

In Fig. 2, $a^2 = h^2 + (c - p)^2$

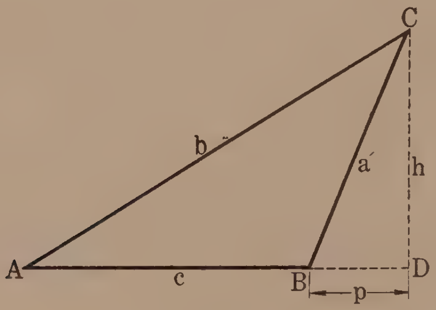
Substitute for h^2 and simplify.

Q.E.D.

PROBLEMS

1. In figure 1 above, if $AC = 12$, $AB = 5$, and $BD = 2$, find BC.
2. In the same figure if $AC = 12$, $AB = 5$, and $CB = 9$, find BD.
3. In the same figure if $AC = 12$, $CB = 10$, and $CD = 8$, find AB.

414. **Theorem.**—In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of those sides and the projection of the other upon that side.



Given the obtuse angle ABC in the $\triangle ABC$ and BD the projection (p) of BC upon AB.

To prove that $b^2 = a^2 + c^2 + 2cp$.

Proof.—$b^2 = h^2 + (c + p)^2$. (1) Why?

Also $h^2 = a^2 - p^2$. (2) Why?

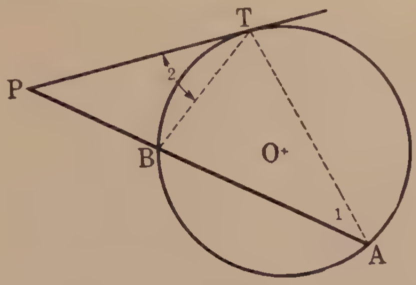
Substituting the value of h^2 from (2) in (1) and simplifying, $b^2 = a^2 + c^2 + 2cp$. *Q.E.D.*

415. **Corollary.**—An angle of a triangle is right, acute or obtuse, according as the square of the side opposite the angle is equal to, is less than, or is greater than the sum of the squares of the other two sides.

PROBLEMS

1. Using the above figure find AC if $AB = 10$, $BC = 12$, and $AD = 14$.
2. Find AB if $AC = 20$, $CD = 10$, and $BC = 14$.
3. What kind of triangles are the following? (a) if the sides are 8, 12, and 15; (b) If the sides are 8, 12, and 14; (c) If the sides are 5, 12 and 13; (d) If the sides are 20, 21 and 29.

416. Theorem.—If from an external point a tangent and a secant are drawn to a circle, the tangent is a mean proportional between the whole secant and its external segment.

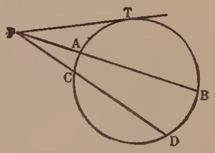


Given the circle O, PT a tangent, and PA a secant.
To prove that $PA:PT = PT:PB$.
Proof.—Draw BT and TA.
Then $\triangle$s PBT and PAT are similar.

For $\angle P$ is common to both and $\angle 1 = \angle 2$ (measured by $\frac{1}{2}\overset{\frown}{BT}$).
$\therefore PA:PT = PT:PB$. Why?

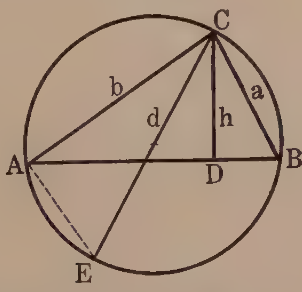
Q.E.D.

PROBLEMS



Use the Fig. of §416 for problems 1, 2, and 3.
1. If $PA = 25$ and $PB = 9$, find PT.
2. If $PT = 10$ and $AB = 20$, find PB.
3. If $PT = 5\sqrt{3}$ and $AB = 2PB$, find PB.
4. Show that $PA \times PB = PC \times PD$ if PT is a tangent, PB and PD are secants.

417. Theorem.—In any triangle the product of any two sides is equal to the product of the diameter of the circumscribed circle and the perpendicular drawn to the third side.



Given $\triangle ABC$ inscribed in the circle ABC. $CD \perp AB$, CE is a diameter.
To prove that $ab = dh$.
Proof.—Draw AE.
Then $\angle E = \angle B$. §277.

Hence, rt. △s CDB and AEC are similar. §311.

$\therefore\ a:d = h:b.$ §307.

$\therefore\ ab = dh.$ Why?

Q.E.D.

PROBLEMS

1. If $a = 10$, $b = 12$, and $d = 15$, find h.
2. If $h = 9$, $a = 10$, and $d = 15$, find b.
3. If $BD = 3$, $a = 5$, and $d = 10$, find b.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

418. **First and Second Degree Equations.**—The simultaneous equations solved previous to this involved only equations of the first degree. We will now solve simultaneous equations in which one or both equations are of the second degree.

419. **Homogeneous Equations.**—A homogeneous (composed of similar parts) equation is one in which the sum of the exponents of the unknowns in each term is the same.

420. **Solution of Simultaneous Equations Involving Quadratics.**—Simultaneous equations involving quadratics may be composed of different combinations of equations.

Illustrations.—I. Solve
$$x + y = 8,$$
$$x^2 + y^2 = 34.$$

Solution.
$$x + y = 8, \tag{1}$$
$$x^2 + y^2 = 34. \tag{2}$$

From (1)
$$y = 8 - x. \tag{3}$$

Squaring (3), $y^2 = 64 - 16x + x^2.$

Substituting for y^2 in (2), $x^2 + 64 - 16x + x^2 = 34.$

Solving,
$$2x^2 - 16x + 30 = 0.$$
$$x^2 - 8x + 15 = 0.$$
$$(x - 5)(x - 3) = 0.$$
$$\therefore\ x = 5. \quad x = 3.$$

Substituting the values of x in (3), $y = 8 - 5$ or 3.

And $y = 8 - 3$ or 5.

Ans.
x, y.
3, 5.
5, 3.

Check both sets of roots.

II. Solve
$$2x^2 + y^2 = 17,$$
$$x^2 - y^2 = -5.$$

Solution.
$$2x^2 + y^2 = 17, \tag{1}$$
$$x^2 - y^2 = -5. \tag{2}$$

By addition,
$$3x^2 = 12.$$
$$x^2 = 4.$$
$$\therefore\ x = \pm 2.$$

Substituting for x in (1), $8 + y^2 = 17.$

$$y^2 = 9.$$

$$\therefore y = \pm 3.$$

$\therefore$ When $x = +2,\quad y = \pm 3.$

And $x = -2,\quad y = \pm 3.$

Here we have four possible sets of values satisfying the equations.

$x = 2,\ y = 3.$

$x = 2,\ y = -3.$

$x = -2,\ y = 3.$

$x = -2,\ y = -3.$

Check each set of values. These results may be written in this form:

x,	y.
2,	± 3.
-2,	± 3.

Note.—*These results cannot be put in the form,* $x = \pm 2$, $y = \pm 3$ *for they include only the first and fourth sets of values.*

III. Solve $x^2 - 2xy = -8,$

$x^2 + y^2 = 25.$

Solution.

$$x^2 - 2xy = -8, \tag{1}$$

$$x^2 + y^2 = 25. \tag{2}$$

Substituting vx for y in (1), $x^2 - 2vx^2 = -8.$

Factoring, $x^2(1 - 2v) = -8.$

$$\therefore x^2 = \frac{-8}{1 - 2v}. \tag{3}$$

Substituting vx for y in (2), $x^2 + v^2x^2 = 25.$

Factoring, $x^2(1 + v^2) = 25.$

$$\therefore x^2 = \frac{25}{1 + v^2.} \tag{4}$$

By comparing (3) and (4), $\dfrac{-8}{1 - 2v} = \dfrac{25}{1 + v^2}$

Clearing of fractions, $-8 - 8v^2 = 25 - 50v.$

Equating to zero, $8v^2 - 50v + 33 = 0.$

$$\therefore v = \tfrac{3}{4} \text{ or } \tfrac{11}{2}.$$

Substituting the values of v in (3), $x^2 = \dfrac{-8}{1 - \frac{8}{2}}$ and $\dfrac{-8}{1 - 11}.$

$$\therefore x^2 = 16 \text{ and } \tfrac{4}{5}.$$

$$\therefore x = \pm 4 \text{ and } \pm \tfrac{2}{5}\sqrt{5}.$$

The same result will be obtained if the values of **v** are substituted in equation (4).

In solving for y, in the equation $y = vx$, be careful to multiply each pair of values of x by the corresponding value of v.

Substituting then in $y = vx$ we have,

When $x = +4,\ y = +3.$

$x = -4,\ y = -3.$

$x = +\frac{2}{5}\sqrt{5},\ y = +\frac{11}{5}\sqrt{5}.$

$x = -\frac{2}{5}\sqrt{5},\ y = -\frac{11}{5}\sqrt{5}.$

Ans.

x,	y
± 4,	± 3,
$\pm \frac{2}{5}\sqrt{5}$,	$\pm \frac{11}{5}\sqrt{5}$.

IV. Solve $x + y = 9,$

$xy = 20.$

Solution (*a*).

$$x + y = 9, \tag{1}$$
$$xy = 20. \tag{2}$$

From (2) $$x = \frac{20}{y}. \tag{3}$$

By substituting for x in (1), $\dfrac{20}{y} + y = 9.$

Solving, $20 + y^2 = 9y.$ *Ans.*

$y^2 - 9y + 20 = 0.$ $x, y.$

$y = 4$ or $5.$ 4, 5.

Substituting for y in (2), $x = 5$ or $4.$ 5, 4.

Solution (*b*).

$$x + y = 9. \tag{1}$$
$$xy = 20. \tag{2}$$

Squaring (1), $x^2 + 2xy + y^2 = 81.$

Multiplying (2) by 4, $4xy = 80.$

Subtracting, $x^2 - 2xy + y^2 = 1.$

Extracting the square root, $x - y = \pm 1.$ (3)

$x + y = 9.$ (1)

$x - y = \pm 1.$ (3)

Ans.

Adding (1) and (3), $2x = 10$ or $8.$ $x, y.$

$\therefore x = 5$ or $4.$ 4, 5.

Subtracting (3) from (1), $2y = 8$ or $10.$ 5, 4.

$\therefore y = 4$ or $5.$

Miscellaneous Problems.

V. Solve $x^3 - y^3 = 37,$ (1)

Solution $x^2 + xy + y^2 = 37.$ (2)

Dividing (1) by (2), $x - y = 1.$ (3)

Squaring (3), $x^2 - 2xy + y^2 = 1.$ (4)

Subtracting (4) from (2), $3xy = 36.$ (5)

$\therefore xy = 12.$ (6)

Adding (2) and (6), $x^2 + 2xy + y^2 = 49.$

$\therefore x + y = \pm 7.$ (7)

$x - y = 1.$ (3)

Adding (3) and (7), $2x = 8$ or $-6.$ *Ans.*

$\therefore x = 4$ or $-3.$ $x, y.$

Subtracting (3) from (7), $2y = 6$ or $-8.$ 4, 3.

$\therefore y = 3$ or $-4.$ $-3, -4.$

VI. Solve

$$x^2 - xy + 3y^2 = 15, \tag{1}$$
$$x^2 - 3xy + y^2 = -5. \tag{2}$$

Solution $\qquad x^2 - xy + 3y^2 = 15. \qquad (1)$

Multiplying (2) by 3, $3x^2 - 9xy + 3y^2 = -15. \qquad (3)$

Adding, $\qquad 4x^2 - 10xy + 6y^2 = 0. \qquad (4)$

Dividing by 2, $\qquad 2x^2 - 5xy + 3y^2 = 0. \qquad (5)$

Factoring, $\qquad (2x - 3y)(x - y) = 0. \qquad (6)$

Solving, $\qquad x = \frac{3y}{2}. \qquad (7)$

$$x = y. \qquad (8)$$

Substituting $\frac{3y}{2}$ for x in (1)

We have, $\frac{9y^2}{4} - \frac{3y^2}{2} + \frac{3y^2}{1} = 15.$

Solving, $\qquad 15y^2 = 60.$

$$y^2 = 4.$$

$$\therefore y = \pm 2.$$

Substituting ± 2 for y in (7),

$$x = 3 \text{ when } y = 2,$$

$$\text{and } x = -3 \text{ when } y = -2.$$

Substituting y for x in (1)

We have, $\qquad y^2 - y^2 + 3y^2 = 15.$

Solving, $\qquad 3y^2 = 15.$

$$y^2 = 5.$$

$$\therefore y = \pm\sqrt{5}.$$

Ans.

x,	y.
± 3,	± 2.
$\pm\sqrt{5}$,	$\pm\sqrt{5}$.

Substituting $\pm\sqrt{5}$ for y in (8),

$$x = \sqrt{5} \text{ when } y = \sqrt{5},$$

$$\text{and } x = -\sqrt{5} \text{ when } y = -\sqrt{5}.$$

Checking we find each of these four sets of values satisfy the equations.

Note.—(*a*) *When one equation is of the first degree and the other equation is of the second degree, the method by substitution can always be used in solving the equations.*

(*b*) *When both equations are of the second degree, the method* $y = vx$, *used in the solution of III, can always be used in solving the equations.*

PROBLEMS

Solve for x and y and check.

1. $x^2 + y^2 = 13, \quad x^2 - y^2 = -5.$
2. $x - y = 1, \quad xy = -\frac{2}{9}.$
3. $x^2 + y^2 = 40, \quad x + y = 4.$
4. $x + y = 16, \quad xy = 63.$
5. $x - y = -12, \quad xy = -35.$
6. $x + y = 5, \quad x^2 + y^2 = 97.$
7. $x - y = 8, \quad x^2 + y^2 = 34.$
8. $2x - 3y = -5, \quad 2x^2 - 3y^2 = -19.$
9. $x + 9y = -22, \quad xy = -15.$
10. $3x - y = 2, \quad x^2 - xy = -4.$
11. $3x - 2y = 11, \quad 3x^2 + 2xy = 52.$
12. $\frac{1}{x} + \frac{1}{y} = \frac{7}{12}, \quad \frac{1}{xy} = \frac{1}{12}.$
13. $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}, \quad \frac{1}{x^2} + \frac{1}{y^2} = \frac{13}{36}.$
14. $x^2 - xy + y^2 = 7, \quad x + y = 5.$

15. $x - y = 15,$ $x^2 - y^2 = 45.$

16. $x^3 + y^3 = 91,$ $x + y = 7.$

17. $x^3 - y^3 = 61,$ $x - y = 1.$

18. $x^3 + y^3 = 117,$ $x + y = 3.$

19. $x^3 - y^3 = 37,$ $x - y = 1.$

20. $x^3 + y^3 = 468,$ $x + y = 12.$

21. $x^2 + 2xy = -3,$ $x - y = 5.$

22. $x^3 + y^3 = 7,$ $x^2 - xy + y^2 = 7.$

23. $x^3 - y^3 = 56,$ $x^2 + xy + y^2 = 28.$

24. $2x^2 - 3xy = -4,$ $4xy - 5y^2 = 3.$

25. $x^2 + 3y^2 = 28,$ $x^2 + xy + 2y^2 = 16.$

26. $y^2 + xy = 3,$ $xy + x^2 = -2.$

27. $3x^2 - 2xy = 39,$ $x^2 + y^2 = 13.$

28. $xy = 20,$ $x^2 + y^2 = 41.$

29. $xy = 16,$ $x^2 + y^2 = 68.$

30. $x^2 + xy + y^2 = 39,$ $x - y = 12.$

31. $x^3 + y^3 = 9a^3,$ $x + y = 3a.$

32. $m^2 + mn + n^2 = 39,$ $2m^2 + 3mn + n^2 = 63.$

33. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{25}{144},$ $\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$

34. $\frac{1}{x} + \frac{1}{y} = \frac{17}{6},$ $xy = \frac{1}{2}.$

35. $x^2 - 2xy = 15,$ $2xy + y^2 = -5.$

36. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2},$ $x + y = 3.$

37. $\frac{y}{x} + \frac{x}{y} = \frac{5}{2},$ $2x - 3y = -5.$

38. $x^2 - y^2 = a^2 - b^2,$ $xy = ab.$

39. $2x^2 + 3y + 10x = -3,$ $x - y = -5.$

40. $y^2 = 4x,$ $x^2 + y^2 = 117.$

41. $2x^2 - 3xy - 2y^2 = 0,$ $3x^2 - 4xy - 4y^2 = 0.$

42. $x^2 + y^2 = 5a^2,$ $xy = 2a^2.$

43. $x^2 + xy = 60,$ $y^2 + xy = 84.$

44. $x^3 + y^3 = 152,$ $x + y = 8.$

45. $2x^2 + 2y - 2x = 72,$ $y + x = 1.$

46. $x^2 - y^2 - 3x = -38,$ $y - 5x = 1.$

47. $x^4 - y^4 = 65,$ $x^2 + y^2 = 13.$

48. The area of a rectangle is 90 square feet and its perimeter is 42 feet. Find its dimensions.

49. The difference of two numbers is 15 and their product is -54. Find the numbers.

50. The sum of two numbers is 9 and the difference of their squares is 9. Find the numbers.

51. The area of a rectangle is 108 square feet and the length of its diagonal is 15 ft. Find its dimensions.

52. The hypotenuse of a right triangle is 15 and the area is 54. Find the base and altitude.

53. If the product of two numbers be multiplied by their sum, the result is 30, and the sum of their cubes is 35. Find the number.

Hint.—*Refer to illustrations V and VI.*

54. The area of a right triangle is 15 and the hypotenuse is $\sqrt{61}$. Find the legs.

55. The perimeter of a right triangle is 48 feet and its area is 96 square feet. Find the hypotenuse and legs of the triangle.

56. The sum of the cubes of two numbers is 189, and their sum is 9. Find the numbers.

57. The ratio of two numbers is 3:2 and their product is 600. Find the numbers.

58. The difference of two numbers is 3, and the sum of their reciprocals is $\frac{13}{40}$. Find the numbers.

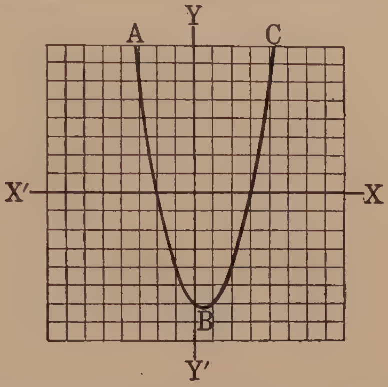
59. Two numbers are in the ratio of 7:4. If 15 be added to both numbers the difference of their squares is 1275.

421. Graph of a Quadratic Equation in Two Unknowns.—Statistical graphs as well as graphs of equations of the first degree in two unknown quantities have been studied. The latter were found to be straight lines. Now we will construct graphs of equations of the second degree in two unknown quantities.

Illustrations.—1. Construct the graph of $y = x^2 - x - 6$.

Solution.—Assign values to x and solve for the corresponding values of y.

x	y
0	-6
$\frac{1}{2}$	$-6\frac{1}{4}$
1	-6
2	-4
3	0
4	6
-1	-4
-2	0
-3	6



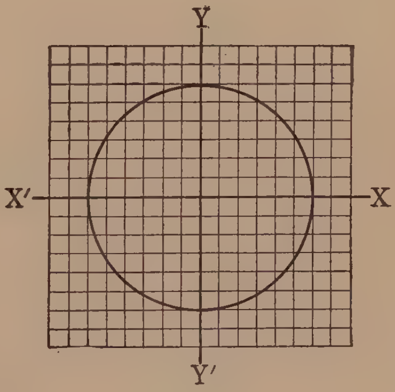
The graph obtained is the curve ABC and is called a **parabola.** The path of a projectile, such as a cannon ball shot through the air, neglecting the friction of the air, is an arc of a parabola in a different position with respect to the axes. This can be proven by analytical geometry.

2. Construct the graph of $x^2 + y^2 = 36$.

Solving for y in terms of x, $y = \pm\sqrt{36 - x^2}$.

Construct the table as in (1).

x	y
0	± 6
1	± 5.9
2	± 5.6
3	± 5.2
4	± 4.5
5	± 3.3
6	0
-1	± 5.9
-2	± 5.6
-3	± 5.2
-4	± 4.5
-5	± 3.3
-6	0



The graph is a circle with its center at the origin and a radius equal to 6.

PROBLEMS

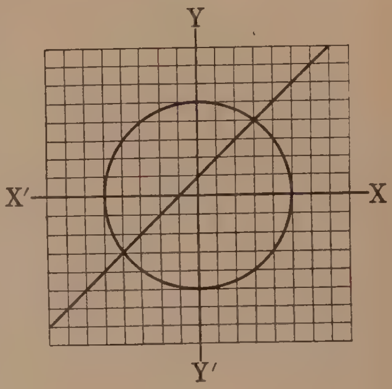
Construct the graphs of the following:

1. $y = x^2 - x - 2$.
2. $x^2 + y^2 = 16$.
3. $4x^2 + 9y^2 = 36$.
4. $x = y^2 - 3y + 5$.
5. $9x^2 - 16y^2 = 144$.
6. $x^2 + y^2 - 4 = 5$.

422. Graphic Solution of Simultaneous Equations Involving Quadratics.

Illustration.—Solve graphically $x - y = -1$ and $x^2 + y^2 = 25$.

Construct the graphs of the two given equations, one of the first degree and the other of the second degree.



As will be seen from the graphs, the line and the circle intersect in the points (3, 4) and (−4, −3).

This is the graphic solution of the two equations. If the equations are solved algebraically we obtain the same result.

Note.—*When the result is written as follows:*

x,	y
3,	4
−4,	−3.

Each pair of values denotes the coordinates of a point of intersection of the graphs of the two equations.

423. Imaginary Roots.—Construct the graphs of $x^2 + y^2 = 9$ and $x + y = 5$. It will be found that the graphs do not intersect. If these two equations are solved, the values of x and y will be imaginary. Thus, the imaginary roots correspond to points that cannot be plotted since the graphs of the given equations do not intersect.

424. Number of Roots.—In general the number of roots in a pair of simultaneous equations is equal to the product of the degree of the two equations. If we have two equations each of the second degree, the graphs of these equations intersect in four points either real or imaginary. The student should illustrate this by constructing the graphs of $y = x^2 - 6$ and $x^2 + y^2 = 9$.

There are, however, some exceptions to the above statement. Thus, in solving the equations $x^3 + y^3 = 27$ and $x + y = 3$, we get only two sets of roots. Illustrate this fact by the graphs of the two equations.

PROBLEMS

Solve by graphs and check.

1. $y^2 = 3x$,
   $x - y = 0$.
2. $x^2 + y^2 = 25$,
   $x - 2y = -5$.
3. $xy = 4$,
   $x + y = 3$.
4. $x^2 + y^2 = 13$,
   $xy = 6$.
5. $4x^2 + 9y^2 = 145$,
   $x^2 + y^2 = 25$.
6. $x^2 + y = 7$,
   $x + y^2 = 11$.
7. $4x^2 - 9y^2 = 36$,
   $2x - 3y = 5$.
8. $3x^2 + y^2 = 3$,
   $x - y = -2$.
9. $x^2 + y^2 = 100$,
   $3x - 4y = 12$.
10. $y^2 = 2x$,
    $xy = 2$.

CHAPTER XII

MENSURATION

425. **Practical Use of Plane and Solid Geometry.**—Engineers are often required to compute areas of geometric figures, such as the surface of floors and walls; the curved surface of cylinders and spheres; the area of a section of land, etc. They must also be able to compute the volumes of geometric solids such as a rectangular bin or a cylindrical tank. In railway construction, the volume of earth to be removed in making a cut or fill is a problem of frequent occurrence. In heating and ventilating the amount of air space in a room is a factor to be considered.

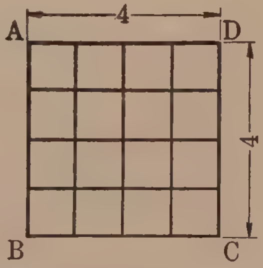
In any of the above problems a knowledge of formulas derived in **plane** and **solid geometry** is required. No attempt will be made to study the derivation of the formulas based on solids, although the more important formulas derived from plane geometry are more fully developed.

The application of geometric formulas most frequently needed will be shown in this chapter by solving problems requiring their use. If the student desires a rigorous analysis and proof of any formula the proof of which is omitted he should consult standard text books on Plane and Solid Geometry.

AREAS OF POLYGONS

426. **Area.**—The area of any surface is the number of square units into which the surface may be divided.

427. **Area of a Square.**—Study the square *ABCD* whose surface is divided into 16 smaller squares.



1. Each dimension *AB* and *BC* is divided into 4 units of the same length.

2. Each little square is called a unit of measure.

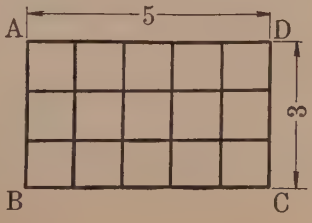
3. The area of each small square is $1 \times 1 = 1$.

4. The area of the large square is the product of the two linear dimensions, thus $4 \times 4 = 16$.

Hence, the area of a square is equal to the square of a side.

Formula.—$A = a^2$. (A = area and a = side.)

428. **Area of a Rectangle.**—By the method of §427, find the area of the rectangle *ABCD* as follows:



1. The length AD is divided into 5 units.
2. The width AB is divided into 3 units.
3. By drawing the lines shown, the rectangle is divided into 15 square units, thus $5 \times 3 = 15$.

Hence, to find the area of a rectangle, multiply the length by the width.

Formula.—$\mathbf{A = lw}$ or $\mathbf{A = bh}$.

PROBLEMS

Find the area of the following:

1. A square 15 inches on a side.
2. A square 25 inches on a side.
3. A rectangle whose dimensions are 16″ and 27″.
4. A rectangle whose dimensions are 20″ and 30″.
5. A rectangle whose dimensions are 54″ and 16″.
6. Find the altitude of a rectangle, if the area is 300 square inches and the base is 15 inches.

Solution.—$A = bh$. where b = the base and h = the altitude. Substituting, $300 = 15h$. $\therefore h = 20$.

7. Find the altitude of a rectangle, if the area is 189 square feet and the base is 9 feet.
8. Find the base of a rectangle whose area is 504 and whose altitude is 12.
9. Find the area of a rectangle, if the base is 21 inches and the altitude is $10\frac{1}{2}$ inches.
10. Find the base of a rectangle, if the altitude is 19 inches and the area is 380 square inches.

429. **Area of a Parallelogram.**—The area of a parallelogram is equal to the product of its base and altitude.

If we think of a parallelogram as a rectangle which has been so forced that its right angles are now all oblique, we can readily grasp the nature of its construction.

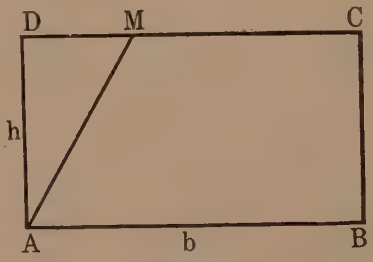


FIG. 1.

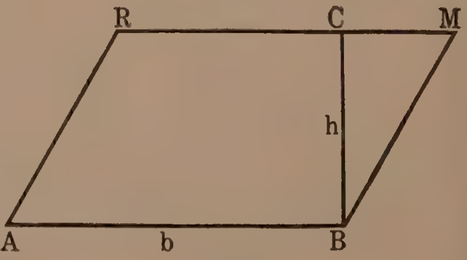


FIG. 2.

Consider first the rectangle $ABCD$, Fig. 1, with AM drawn forming the rt. △ AMD. Now remove the rt. △ AMD and let it take the position of the △ BMC in Fig. 2.

Then,

1. The bases **b** and the altitudes **h** of both figures remain unchanged.
2. The areas remain unchanged, for the △ cut from Fig. 1 is placed in another position to form Fig. 2.
3. The area of the rectangle in Fig. 1 is **bh**.

4. The area of the parallelogram in Fig. 2 must also be **bh.**

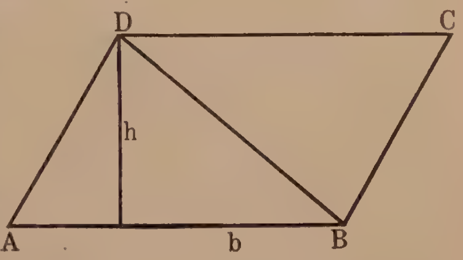
Formula.—A = bh.

PROBLEMS

Find the area of a parallelogram whose base and altitude are:

1. 10 and 20.
2. 14 and 22.
3. 16 and 19.95.
4. 82 and 21.63.

430. **Area of a Triangle.**—The area of a triangle is equal to one-half the product of its base and altitude.



Given the parallelogram *ABCD* which is divided into two triangles by the diagonal *BD*.

Then,

1. The diagonal of a parallelogram divides it into two equal triangles. §154.

2. The altitude **h** and the base **b** are the altitude and base for both the triangle and the parallelogram.

3. The area of the parallelogram is **bh.**

4. Since the area of each triangle is one-half the area of the parallelogram, the area of the triangle is $\frac{1}{2}bh$.

Formula.—A = $\frac{1}{2}$**bh.**

PROBLEMS

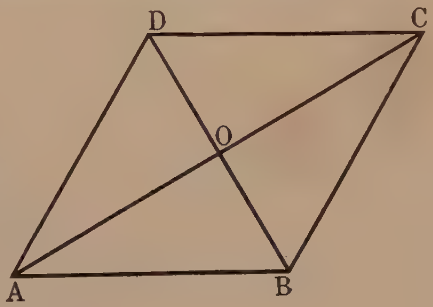
Find the area of a triangle, having given:

1. $b = 10$, $h = 12$.
2. $b = 25$, $h = 20$.
3. $h = 16$, $b = \frac{1}{2}$.
4. $h = 5.5$, $b = 10.2$.

Find the unknown in the following using the same formula.

5. $A = 120$, $b = 24$.
6. $h = 10.7$, $A = 62.5$.
7. $A = 35.5$, $b = 20$.
8. $A = 490$, $b = 28$.

431. **Area of a Rhombus.**—The area of a rhombus is equal to one-half the product of its two diagonals.



Given the rhombus *ABCD* whose diagonals are *AC* and *DB*.

Then,

1. The diagonals of a rhombus are perpendicular to each other. *DO* is then an altitude of the $\triangle$ *ACD* on *AC* as a base, and *OB* is an altitude of the $\triangle$ *ABC* on *AC* as a base.

2. The area of $\triangle ACD = \frac{1}{2} AC \times OD$.

3. The area of $\triangle ABC = \frac{1}{2} AC \times OB$.

4. By addition, the area of a rhombus $ABCD = \frac{1}{2} AC(OD + OB)$, or $\frac{1}{2} AC \times BD$. *Ax.* 12.

Formula.—A = $\frac{1}{2}d \cdot d_1$. (d and d_1 are the diagonals.)

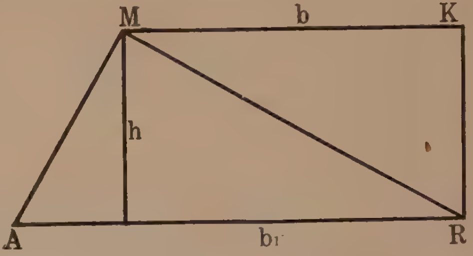
PROBLEMS

Find the area of a rhombus if the diagonals are:

1. 18 and 10.
2. 32 and 12.25.
3. 16 and 24.
4. 48 and 22.08.

432. **Area of a Trapezoid.**—The area of a trapezoid is equal to one-half the product of the altitude and the sum of its bases.

Given the trapezoid *ARKM* in which the lower base is $\mathbf{b_1}$, the upper base is **b**, the altitude is **h**, and the diagonal is *MR*.



Then,

1. The area of $\triangle ARM = \frac{1}{2} hb_1$.
2. The area of $\triangle MRK = \frac{1}{2} hb$.
3. By addition, the area of the trapezoid $ARKM = \frac{1}{2} h(b + b_1)$. *Ax.* 12.

Formula.—$\mathbf{A = \frac{1}{2} h(b + b_1)}$.

PROBLEMS

Find the area of a trapezoid whose bases and altitude are:

1. $b = 10$,
   $b_1 = 14$,
   $h = 15$.
2. $b = 3.5$,
   $b_1 = 6.5$,
   $h = 20$.
3. $b = 3.58$,
   $b_1 = 5.22$,
   $h = 12.32$.

Note.—*Since the median of a trapezoid is equal to one-half the sum of its bases, the formula* $A = \frac{1}{2}h(b + b_1)$, *may be changed to* $A = hm$, *where* m *is the length of the median.* §164.

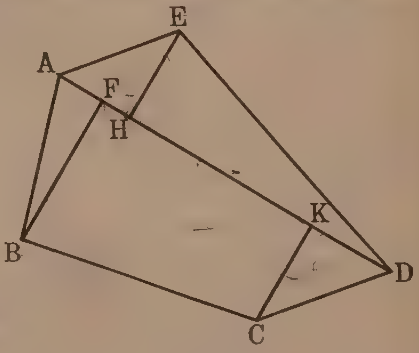
MISCELLANEOUS PROBLEMS

1. A surveyor in measuring the area represented by the polygon *ABCDE* divides it into right triangles and trapezoids as shown. *BF*, *EH*, and *CK* are all perpendicular to the diagonal *AD*.

The following data, in feet, are computed from field notes.

$AF = 25.$
$BF = 70.$
$HK = 105.$
$KC = 55.$
$KD = 45.$
$HE = 50.$
$FH = 15.$

Compute the area of the polygon.



PROB. 1.

2. Compute the area of the polygon *ABCDEG*, having given the following dimensions, in feet.

$MC = 75.$
$HD = 170.$
$MB = 75.$
$BH = 35.$
$HA = 45.$
$AJ = 90.$
$JK = 50.$
$KE = 115.$
$JG = 25.$

Note.—(*a*) *The figure MCDEK = trapezoid MCDH + trapezoid HDEK.*

(*b*) *From this total subtract (rt.* $\triangle$ *MCB + rt.* $\triangle$ *AGJ + trapezoid JGEK) and there remains the polygon, HBCDEGA whose area is to be found.*

3. Find the area of a city lot $ABCD$ bounded by two parallel streets AB and DC. A and D are right angles, AB is 120 ft., DC is 130 ft., and AD is 45 ft.

4. In the figure of problem 3, draw diagonal BD and compute the area by finding the areas of the two triangles. Does it check with problem 3?

5. The area of a triangle is 150 sq. ft. The ratio of the altitude to the is base 2:3. Find its dimensions. (Use $2x$ and $3x$ for altitude and base.)

6. Find the area of a right triangle whose base is 16 and whose altitude is 22.

7. Find the area of a rectangle whose base is 16 and whose diagonal is 20.

8. Find the area of a trapezoid whose bases are 16 and 20 and whose altitude is 12.

9. Find the side of a square whose area equals that of a rectangle whose base is 9 and altitude 4.

10. Find the altitude of a trapezoid whose area is 120 and whose bases are 11 and 15.

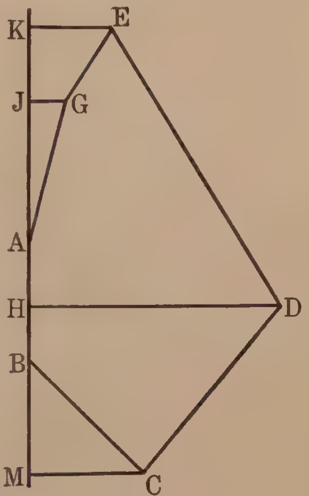
11. Find the area of a square whose perimeter is 44 feet.

12. The area of a trapezoid is 140 and the sum of its bases is 28. Find the altitude. If one base is 6 more than the other, find the bases.

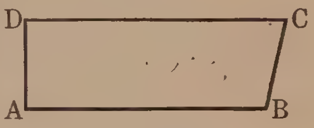
13. Find the side of a square whose area is equal to that of a triangle whose base is 16 and altitude 8.

14. Find the side of a square whose area is equal to that of a parallelogram whose base is 16 and whose altitude is 9.

15. Find the area of a rectangle whose base and altitude are 40 and 30 respectively.



PROB. 2.



PROB. 3.

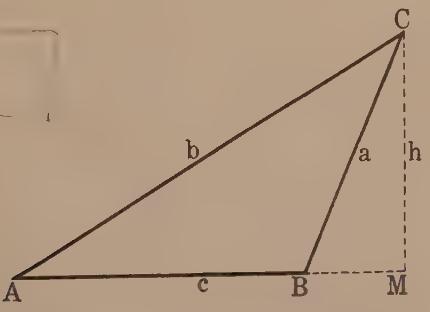
16. Find the area of a rhombus whose diagonals are 20 and 24.

17. Find the area of a rhombus, the sum of whose diagonals is 44 and their ratio is 5:6.

18. Find the area of a trapezoid whose altitude is 10 and median 13.

19. Find the area of a rhombus whose perimeter is 40 and altitude 8.

433. To Compute the Altitude of Any Triangle When the Three Sides are Given.



Given the △ ABC with the sides denoted by **a**, **b**, and **c** and ∠ A acute. Altitude **h** is drawn to side **c** extended.

In △ CAM, $h^2 = b^2 - \overline{AM}^2$. (1)

In △ ABC, $a^2 = b^2 + c^2 - 2c \times AM$.

Solving for AM,

$$AM = \frac{b^2 + c^2 - a^2}{2c}.$$

Substituting in (1),

$$h^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}.$$

Factoring,

$$h^2 = \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2}.$$

$$= \frac{[(b + c)^2 - a^2][a^2 - (b - c)^2]}{4c^2}.$$

$$= \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4c^2}.$$

Rearranging factors,

$$h^2 = \frac{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}{4c^2}. \qquad (2)$$

Now let $a + b + c = 2s$.

Then $-a + b + c = 2(s - a)$.

$a - b + c = 2(s - b)$.

$a + b - c = 2(s - c)$.

Substituting in (2)

$$h^2 = \frac{(2s)2(s - a)2(s - b)2(s - c)}{4c^2}.$$

$$= \frac{16s(s - a)(s - b)(s - c)}{4c^2}.$$

$$= \frac{4s(s - a)(s - b)(s - c)}{c^2}.$$

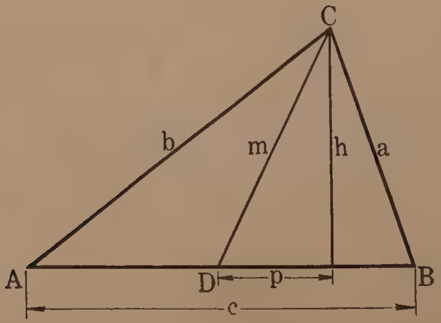
$$\therefore \mathbf{h} = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

We may designate this altitude as $\mathbf{h}_c$.

If **h** is drawn to the side **a** then, $\mathbf{h}_a = \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)}$.

If **h** is drawn to the side **b** then, $\mathbf{h}_b = \frac{2}{b}\sqrt{s(s - a)(s - b)(s - c)}$.

434. To Compute the Median of a Triangle When the Three Sides are Given.



Given the $\triangle ABC$ in which **h** is the altitude and **m** the median drawn to the side **c**, and **p** the projection of **m** on AB.

In $\triangle DBC$, $a^2 = m^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{c}{2}\right)p$. §413.

In $\triangle ADC$, $b^2 = m^2 + \left(\frac{c}{2}\right)^2 + 2\left(\frac{c}{2}\right)p$. §414.

Adding, $a^2 + b^2 = 2m^2 + 2\left(\frac{c}{2}\right)^2$.

$\therefore 4m^2 = 2(a^2 + b^2) - c^2$. §§394, 395.

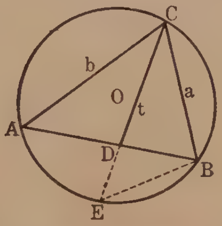
$\therefore m = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$.

We may designate this median as $\mathbf{m_c}$

If **m** is drawn to **b** then $\mathbf{m_b} = \frac{1}{2}\sqrt{2(a^2 + c^2) - b^2}$.

If **m** is drawn to **a** then $\mathbf{m_a} = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}$.

435. To Compute the Bisector of an Angle of a Triangle in Terms of the Sides.



Given the $\triangle ABC$ and **t** the bisector of $\angle C$. Circumscribe a ⊙ about $\triangle ABC$ and extend CD to E, draw EB.

Then $\triangle ADC$ is similar to $\triangle CEB$. §310.

$\therefore CE:b = a:t$. Why?

$\therefore ab = t \cdot CE$.

$ab = t(t + DE) = t^2 + t \cdot DE$.

But $t \cdot DE = AD \cdot DB$. §322.

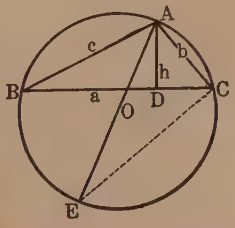
Substituting, $ab = t^2 + AD \cdot DB$.

$\therefore t^2 = ab - AD \cdot DB$.

This can be reduced to the form,

$$\mathbf{t_c} = \frac{2}{a + b}\sqrt{abs(s - c)}.$$

436. To Compute the Radius of a Circle in Terms of the Sides of the Inscribed Triangle.



Given the $\triangle ABC$ inscribed in ⊙ O with AE a diameter.

Draw EC.

Then rt. $\triangle$s AEC and ABD are similar. §311.

$\therefore\ b:h = 2R:c$, where $2R = AE$. (1)

$\therefore\ bc = 2R\cdot h$ or $R = \dfrac{bc}{2h}$. (2)

But $h = \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$. §433.

Substituting in (2), $R = \dfrac{bc}{2\left(\frac{2}{a}\right)\sqrt{s(s-a)(s-b)(s-c)}}$.

Simplifying, $R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$.

MISCELLANEOUS PROBLEMS

1. Find the legs of an isosceles triangle whose base is 80 and whose altitude is $10\sqrt{39}$.

2. Find the legs of an isosceles right triangle whose hypotenuse is $15\sqrt{2}$.

3. The side of an equilateral triangle is **s**. Find the altitude **h**.

4. If the altitude of an equilateral triangle is **h** find the side **s**.

5. The legs of a right triangle are in the ratio of 3:4, the hypotenuse is 60. Find the legs.

6. The sides of a rectangle are 90 and 50. Find the diagonal.

7. The sides of a triangle are 8, 7, and 12. (*a*) Find the three altitudes. (*b*) Find the three medians. (*c*) Find the three bisectors of the angles.

8. A base ball diamond is 90 feet square. Find the diagonal distance between bases.

9. The sides of a triangle are 12, 16 and 20. Find the segments of sides made by the bisectors of the angles opposite. §318.

10. Find the longest and shortest chords that can be drawn through a point 8 inches from the center of a circle whose radius is 12 inches.

11. A diameter is perpendicular to a chord 16 inches long. The radius is 10 inches and one segment of the diameter is 4 inches. Find the distance of the chord from the center.

12. In problem 11 find also the lengths of chords joining the extremities of the diameter to the given chord.

13. In a circle whose radius is 30 inches two parallel chords 56 and 58 inches are drawn. Find the distance between the chords. Is there more than one solution?

14. From one external point two tangents are drawn to a circle of radius 12 inches. The distance from the external point to the circumference is 15 inches. Find the length of the tangents and of the chord joining the points of tangency.

15. The radii of two concentric circles are 8 and 16 inches. Find the length of the chord of the larger circle tangent to the smaller.

16. From an external point two secants are drawn. The segments of the one are 5 and 10, and of the other 6 and x respectively. Find x. (2 solutions.)

17. The distance from the center of a circle to a 12 foot chord is 14 feet. Find the distance from the center to a 10 foot chord.

18. Find the three altitudes of a triangle whose sides are 22.4, 35.6, and 46.2.

19. Find the radius of the circle circumscribed about the triangle of problem 18.

20. Find the radius of the circle circumscribed about a triangle whose sides are 5, 12, and 13.

21. The radii of two intersecting circles are 9 and 12 inches, and half the common chord is 8 inches. Find the line of centers.

22. The radii of two concentric circles are 8 and 12. Find the length of a chord of the larger circle which is tangent to the smaller circle.

23. Find the length of the chord of problem 22 when the radii are x and y.

24. A secant and its external segment are 47 and 15. Find the length of the tangent to the circle from the same external point.

25. The bases of an isosceles trapezoid are 6 and 9 and the altitude is 5.3. Find the legs and the diagonals.

26. The perimeter of a rectangle is 70 and the diagonal is 25. Find the sides.

27. A chord and the radius of a circle are each 10 feet. Find the distance of the chord from the center.

28. In problem 27, if the chord equals $\frac{3}{2}$ of the radius how far is it from the center?

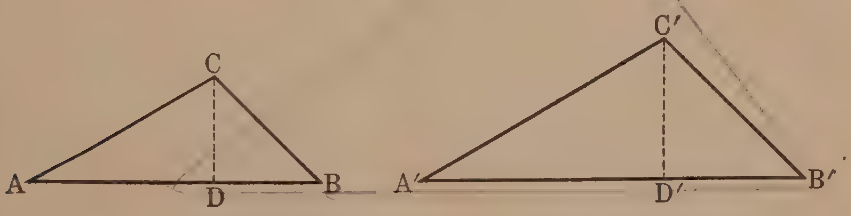
29. The radius of a circle is **r**. What is the length of the chord if it is $\frac{2}{3}r$ distance from the center?

30. The sides of a triangle are 82.4, 101.6, and 132.6. Find the altitude on the side 101.6.

31. The sides of a triangle are 83, 94, 112. Are the angles opposite the respective sides acute, right, or obtuse?

AREAS OF SIMILAR FIGURES

437. **Theorem.**—The areas of two similar triangles are to each other as the squares of any two corresponding sides.



Given the two similar triangles ABC and $A'B'C'$.

To prove that $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Proof.—Draw the altitudes CD and $C'D'$.

Now $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \times CD}{A'B' \times C'D'}$. *Any two △s are to each other as their areas.*

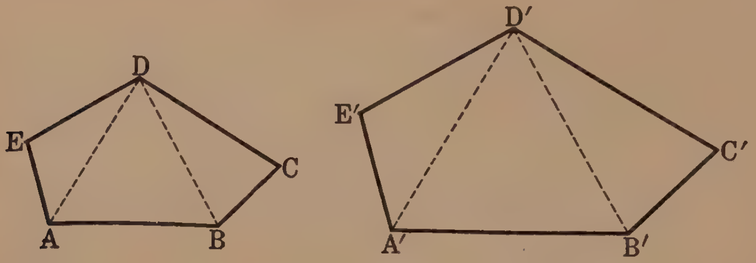
But $\dfrac{AB \times CD}{A'B' \times C'D'} = \dfrac{AB}{A'B'} \times \dfrac{CD}{C'D'}$, and $\dfrac{AB}{A'B'} = \dfrac{CD}{C'D'}$. §316.

Substituting $\dfrac{AB}{A'B'}$ for its equal $\dfrac{CD}{C'D'}$

We have, $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB}{A'B'} \times \dfrac{AB}{A'B'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$. *Q.E.D.*

438. **Corollary.**—The areas of two similar triangles are to each other as the squares of any two corresponding altitudes.

439. **Theorem.**—The areas of two similar polygons are to each other as the squares of any two corresponding sides.



Given the similar polygons $ABCDE$ and $A'B'C'D'E'$ whose areas are denoted by A and A' respectively.

To prove that $A:A' = \overline{AE}^2:\overline{A'E'}^2$.

Proof.—Draw the diagonals DA, DB, $D'A'$, and $D'B'$.

$$\text{Now } \frac{\overline{AE}^2}{\overline{A'E'}^2} = \frac{\triangle\ AED}{\triangle\ A'E'D'} = \frac{\overline{AD}^2}{\overline{A'D'}^2} = \frac{\triangle\ ABD}{\triangle\ A'B'D'} \text{ etc.} \qquad \S 437.$$

$$\text{That is, } \frac{\triangle\ AED}{\triangle\ A'E'D'} = \frac{\triangle\ ABD}{\triangle\ A'B'D'} = \frac{\triangle\ BCD}{\triangle\ B'C'D'}. \qquad Ax.\ 7.$$

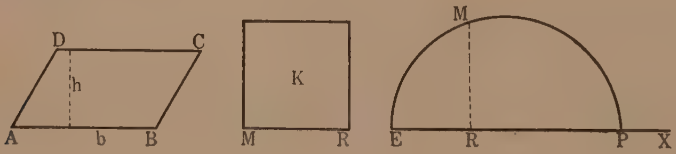
$$\therefore \frac{\triangle\ AED + \triangle\ ABD + \triangle\ BCD}{\triangle\ A'E'D' + \triangle\ A'B'D' + \triangle\ B'C'D'} = \frac{\triangle\ AED}{\triangle\ A'E'D'} = \frac{\overline{AE}^2}{\overline{A'E'}^2}. \qquad \S 301.$$

$\therefore A:A' = \overline{AE}^2:\overline{A'E'}^2$. *Q.E.D.*

440. **Corollary.**—The areas of two similar polygons are to each other as the squares of any two corresponding diagonals.

441. **Corollary.**—The square root of the areas of any two similar polygons are to each other as any two corresponding sides.

442. **Problem.**—To construct a square having the same area as a given parallelogram.



Given the parallelogram $ABCD$, b the base, and h the altitude.

To construct a square equal in area to $\square\ ABCD$.

Construction.—On EX take ER = h and RP = b.

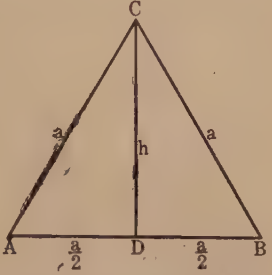
On EP as a diameter describe a semi-circle and draw $MR \perp EP$ cutting the semi-circumference at M.

On MR as a side construct the required square-K.

Proof.—$\overline{MR}^2 = ER \times RP$. §407.

$\therefore$ The square on $MR \simeq \square\ ABCD$. *Q.E.F.*

443. To Compute the Area of an Equilateral Triangle in Terms of Its Sides.—Let a be a side, h an altitude, and A the area.

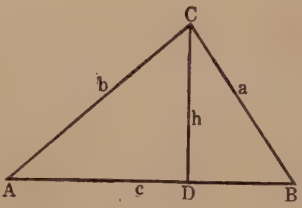


Then $h^2 = a^2 - \frac{a^2}{4}$. $\therefore h = \frac{a}{2}\sqrt{3}$.

But $A = \frac{1}{2}ah$. $\therefore A = \frac{1}{2}a \cdot \frac{a}{2}\sqrt{3}$.

Simplifying, $\mathbf{A} = \frac{\mathbf{a}^2}{4}\sqrt{3}$.

444. To Compute the Area of a Triangle in Terms of its Sides.—Let **h** be drawn to side c.

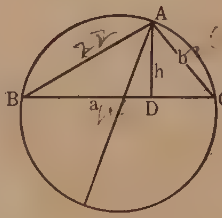


Then $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. §433.

But $A = \frac{c}{2} \times \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. §430.

$\therefore \mathbf{A} = \sqrt{\mathbf{s(s-a)(s-b)(s-c)}}$.

445. To Compute the Area of a Triangle in Terms of Its Sides and the Radius of the Circumscribed Circle.—Let **R** denote the radius of the circumscribed circle and **h** the altitude of the $\triangle ABC$.



Then $bc = 2R \cdot h$. §436–2.

Multiplying by a, $abc = 2R \cdot a \cdot h$.

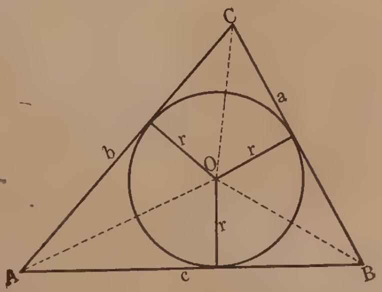
But $ah = 2A$. §430.

$abc = 2R \cdot 2A$ or $4RA$.

$\therefore \mathbf{A} = \frac{\mathbf{abc}}{\mathbf{4R}}$.

It follows that $\mathbf{R} = \frac{\mathbf{abc}}{\mathbf{4A}}$.

446. To Compute the Area of a Triangle in Terms of Its Sides and the Radius of the Inscribed Circle.—Denote the radius of the inscribed circle by **r.**



Area of $\triangle AOB = \frac{1}{2}c \times r$.

Area of $\triangle BOC = \frac{1}{2}a \times r$.

Area of $\triangle AOC = \frac{1}{2}b \times r$.

Adding, $A = \frac{1}{2}(a + b + c)r$.

But $a + b + c = 2s$.

$\therefore \mathbf{A} = \frac{1}{2}(2s)r$ or $\mathbf{sr}$.

447. To Compute the Radius of a Circle Inscribed in a Triangle in Terms of Its Sides.

$$A = sr. \quad \therefore r = \frac{A}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

PROBLEMS

1. A house is 50 feet long, 40 feet wide, and 30 feet high. The ridge pole above this point is 25 feet high and extends the length of the house. Compute the area to be painted including the roof surface.

2. The base and altitude of a rectangle are 12 and 5 feet respectively. Find the side of a square of equal area.

3. Compute the side of a square equivalent to a right triangle whose legs are each 9.5 feet.

4. Find the area of an isosceles right triangle whose hypotenuse is 4.5 feet.

5. Find the area of a triangle whose sides are 7, 15, and 20.

6. (*a*) Find the area of a triangle whose sides are 22, 26, and 40. (*b*) Find the radius of the circumscribed circle. (*c*) The radius of the inscribed circle.

7. Find the area of an equilateral triangle whose side is 20.

8. The altitude and base of a triangle are 16 and 18 respectively. Find the side of an equivalent square.

9. The area of a triangle is 450 square feet and its altitude is 40 feet. Find the base of an equivalent rectangle having the same altitude.

10. The bases of two similar triangles are 6 and 8. The area of the first is 72. Find the area of the second.

11. The base of a triangle is 12 and the altitude is 6. Find the perimeter of a rhombus of equal area, whose altitude is 4.

12. Find the side of a square if the area is equal to that of an equilateral triangle whose side is 12.

13. The sides of a triangle are 9, 15, and 20. Compute A, R, and r.

14. The sides of a triangle are 20, 40, and 30. Compute A, R, and r.

15. The sides of a triangle are 22.5, 32.8, and 48.7. Compute A, R, and r.

16. The corresponding altitudes of two similar triangles are 9 and 12. Compare their areas.

17. The bases of an isosceles trapezoid are 12 feet and 16 feet and the area is 56 square feet. Find the length of the legs.

18. Find the perimeter, area, and diagonals of a rectangle whose dimensions are 90 and 40.

19. The sides of a right triangle are in the ratio of 6:8:10. The hypotenuse is 50. Find the legs.

20. Find the altitude on the hypotenuse of the triangle in problem 19.

21. The sides of two squares are 9 and 7. Find the side of a square equivalent to the sum of the two squares.

22. Find the sides of a square equal in area to a triangle whose altitude is 18 and whose base is 36.

23. Find the area of a rhombus whose diagonals are 4.62 and 3.31.

24. The area of a rhombus is 30. The altitude is 1 foot less than a side. Find the altitude and a side.

25. Find the area of a rhombus if the perimeter is 200 and one diagonal is 80.

26. Two similar triangles contain a total area of 500. Two corresponding sides are in the ratio of 2:1. Find their areas.

27. The area of a rectangle is 1536 sq. ft. The ratio of the sides is 2:3. Find the dimensions.

28. The side of a square is 5. Find the side of a square 9 times as large.

29. The area of two similar polygons are to each other as 3:5. The area of the first is 300. Find the area of the second.

30. The legs of a right triangle are in the ratio of 5:7. The area is 437.5. Find the legs.

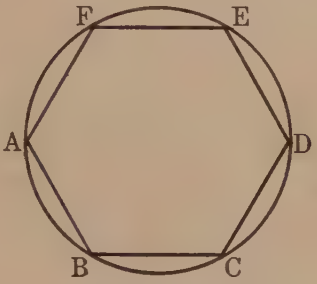
31. The corresponding altitudes of two similar triangles are 8 and 16. Compare their areas. If the area of the first is 200, what is the area of the second?

448. Regular Polygons.—A regular polygon is a polygon whose sides are equal and whose angles are equal. It is equilateral and equiangular.

We have had illustrations of this in the equilateral triangle and the square.

449. Theorem.—Any equilateral polygon inscribed in a circle is a regular polygon.

40



Given *ABCDEF* an equilateral polygon inscribed in a ⊙.

To prove that *ABCDEF* is a regular polygon. (Prove *ABCDEF* equiangular.)

Proof.—The arcs *AB*, *BC*, and *CD*, etc. are equal. §185–5.

Then the arcs *ABC*, *BCD*, etc. are equal. *Ax.* 1.

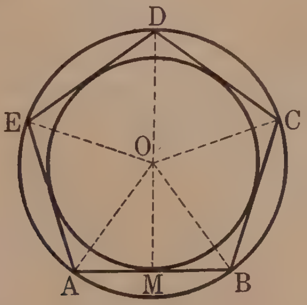
Then the arcs *CEA*, *DFB* are equal. Why?

∴ ∠s *ABC*, *BCD*, etc. are equal. §277.

∴ The polygon is regular, being both equiangular and equilateral.

Q.E.D.

450. Theorem.—A circle may be circumscribed about, or inscribed in, any regular polygon.



Given *ABCDE* a regular polygon.

I. **To prove** that a circle may be circumscribed about *ABCDE*.

Proof.—Pass a circle through *A*, *B*, and *C*. §170.

Draw *OA*, *OB*, *OC*, and *OD*.

Then $\angle ABC = \angle BCD$. Given.
and $\angle OBC = \angle OCB$. §110.
Then $\angle OBA = \angle OCD$. *Ax.* 2.
Now in the $\triangle$s ABO and OCD we have,
$AB = CD$. Given.
$OB = OC$. Why?
$\angle OBA = \angle OCD$. (Proved above)
$\therefore \triangle ABO = \triangle OCD$. §105.
Then $OA = OD$. §104.

$\therefore$ A circle which passes through A, B, and C will also pass through D. In the same way it can be shown that this circle will pass through E. $\therefore$ A circle with center O and radius OA will pass through all the vertices of the polygon.

II. A circle may also be inscribed in this polygon.

Since the polygon is regular and therefore equilateral, the sides are equal chords of the circumscribed circle.

These equal chords are equally distant from the center.

Therefore, a circle with center O and a radius equal to the perpendicular distance from O to one of these equal chords will be inscribed in the polygon. *Q.E.D.*

451. **Radius of a Regular Polygon.**—The radius of a regular polygon is the radius of the circumscribed circle.

452. **Center of a Regular Polygon.**—The center of a regular polygon is the center of the inscribed and circumscribed circles.

453. **Apothem of a Regular Polygon.**—The apothem of a regular polygon is the radius of the inscribed circle.

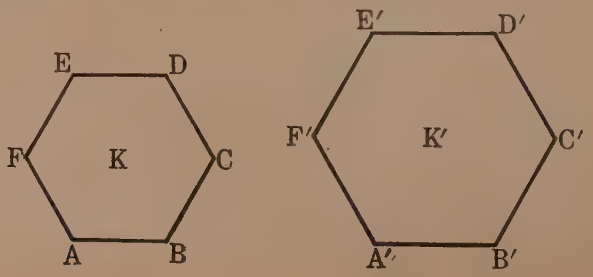
454. **Central Angle of a Regular Polygon.**—The angle at the center of a regular polygon is the angle formed by two radii drawn to the extremities of any side.

455. **Corollary 1.**—The radii of a regular polygon bisect the angles of the polygon.

456. **Corollary 2.**—The angle at the center of a regular polygon is equal to four right angles divided by the number of sides. Therefore all central angles are equal.

457. **Corollary 3.**—An interior angle of a regular polygon, and an angle at the center are supplementary.

458. **Theorem.**—Two regular polygons of the same number of sides are similar.



Given K and K' regular polygons of **n** sides.
To prove that K and K' are similar.

Proof.—Each ∠ of either polygon $= \frac{(n-2)}{n}$ 2 rt. ∠s. §151.

Also $AB = BC = CD$, etc.
And $A'B' = B'C' = C'D'$, etc. §448.
Dividing, $AB:A'B' = BC:B'C' = CD:C'D' = DE:D'E'$, etc.

∴ The polygons are similar, for their corresponding angles are equal and their corresponding sides are proportional. *Q.E.D.*

459. **Corollary 1.**—The perimeters of two regular polygons of the same number of sides are to each other as their apothems or as their radii.

Hint.—*Draw the corresponding apothems and radii in the figures of* §458. *Apply* §319 *and problem* 1, §315.

460. **The Limit of a Variable.**—The limit of a variable is a constant quantity such that the difference between the constant and the variable may become and remain less than any assignable positive quantity however small.

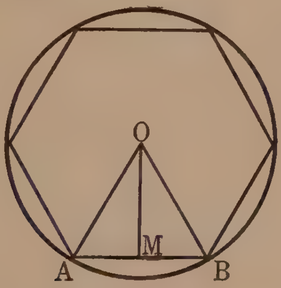
The following illustration will help clarify the idea of a variable approaching a limit.

If the distance between two points P and Q is known, and we walk from P towards Q under the condition that during the first minute we walk one-half the whole distance; during the second minute, one-half the remaining distance, and so on, walking one-half the remaining distance each succeeding minute we will never arrive at Q.

Here the distance PQ is a constant and the distance walked is a variable which never equals PQ. PQ is the limit of the sum of the distances walked.

The variable may increase or decrease towards its limit.

461. **Theorem.**—If the number of sides of a regular inscribed polygon be indefinitely increased, the apothem increases in length and approaches the radius of the circle as its limit.



Given AB a side, OA the radius, and OM the apothem of a regular inscribed polygon of **n** sides.

To prove that OM approaches OA as its limit, when the number of sides of the given polygon is indefinitely increased.

Proof. $(OA - OM) < AM$
and $(OA - OM) < AB$. (For AB is twice AM.)

Think of the number of sides being indefinitely increased.

Then AB can be made less than any assigned value.

But $(OA - OM) < AB$.

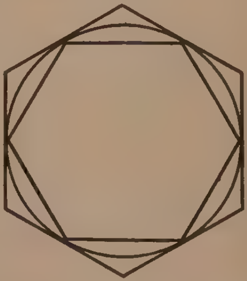
$\therefore$ $(OA - OM)$ can be made less than any assigned value.

$\therefore$ OM tends to lengthen and approach OA as its limit.

PROBLEMS

1. The area of a circle is greater than the area of its inscribed polygon.
2. The area of a circle is less than the area of its circumscribed polygon.

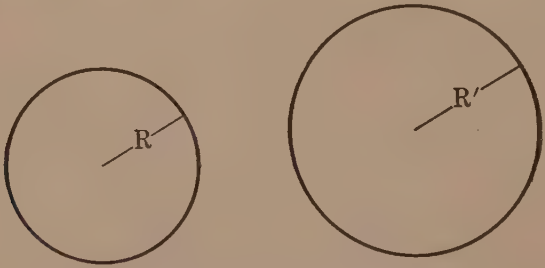
462. **Theorem.**—The circumference of a circle is the limit which the perimeters of the regular inscribed and circumscribed polygons approach, if the number of sides is indefinitely increased, and the area of the circle is the limit which the areas of these polygons approach.



A rigorous proof of this theorem will not be attempted here. By a study of the figure the student will be able to satisfy his needs at this point by picturing the perimeter of the circumscribed polygon as decreasing and the perimeter of the inscribed polygon as increasing as the number of sides of both polygons are indefinitely increased. They will be seen to approach the circumference of the circle as a limit.

At the same time the area of the circumscribed polygon becomes smaller and the area of the inscribed polygon becomes larger and each approaches the area of the circle as its limit.

463. **Theorem.**—The circumferences of two circles are to each other as their radii.



Given R and R' the radii of the circles whose circumferences are C and C'.

To prove that $C:C' = R:R'$.

Proof.—Inscribe in the circles regular polygons of the same number of sides denoting their perimeters by P and P'.

$\therefore$ $P:P' = R:R'$. §459.

Now think of the number of sides of each polygon as being indefinitely increased, but not destroying the similarity of the polygons.

Then the perimeters P and P' will become constantly larger, approaching C and C' as their limits. For our purpose we may substitute C for P and C' for P'. §§460 and 462.

$\therefore C : C' = R : R'$.

464. **Corollary 1.**—The circumferences of two circles are to each other as their diameters, or $C : C' = D : D'$.

465. **Corollary 2.**—The ratio of a circumference to its diameter is constant and is represented by the Greek letter π. For by alternation,

$$\frac{C}{C'} = \frac{D}{D'} \text{ becomes } \frac{C}{D} = \frac{C'}{D'}.$$

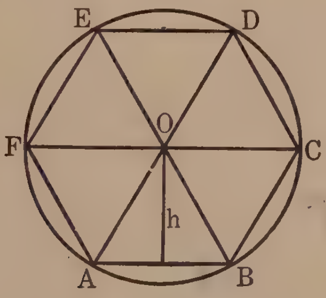
$$\therefore \frac{C}{D} \text{ is the constant ratio represented by } \pi.$$

466. **Corollary 3.**—The circumference of a circle is equal to π times the diameter or two π times the radius.

$$\text{For } \frac{C}{D} = \pi. \quad \therefore C = \pi D \text{ or } 2\pi R.$$

Note. $\pi = 3.1416$. *For practical purposes* $\frac{22}{7}$ *is often used for the value of* π.

467. **Theorem.**—The area of a regular polygon is equal to one-half the product of its perimeter and apothem.



Given p the perimeter, **A** the area, and **h** the apothem of the regular polygon $ABCDEF$.

To prove that $A = \frac{1}{2} hp$.

Proof.—Draw the radii OA, OB, OC, etc.

The polygon is now divided into equal isosceles △s.

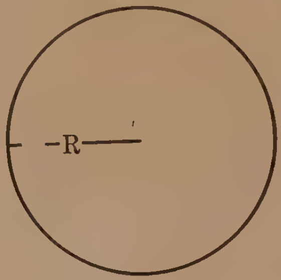
The area of each △ is $\frac{1}{2}h$ times its base for all the △s have equal altitudes (the apothem).

The sum of the bases of all the △s $= p$. The sum of the areas of all △s $= A$.

Hence, $A = \frac{1}{2}h(AB + BC + CD + \text{etc.})$ or $\mathbf{A = \frac{1}{2} hp}$.

Q.E.D.

468. **Theorem.**—The area of a circle is equal to one-half the product of its radius and circumference.



Given R the radius of a circle, **A** the area, and **C** the circumference.

To prove that $A = \frac{1}{2}RC$.

Proof.—Circumscribe a regular polygon about the circle denoting its perimeter by P and its area by A'.

Now $A' = \frac{1}{2}RP$. §467.

Think of the number of sides of this polygon as being increased indefinitely. Then **P** approaches **C** as its limit and A' becomes larger approaching A as its limit.

$\therefore \frac{1}{2}RP$ approaches $\frac{1}{2}RC$ as its limit.

But $A' = \frac{1}{2}RP$.

$\therefore \mathbf{A} = \frac{1}{2}\mathbf{RC}$. (Substituting as in §463.)

469. **Sector.**—A sector of a circle is that portion of a circle bounded by two radii and their intercepted arc.

470. **Corollary 1.**—The area of a circle is equal to π times the square of the radius.

The area of a circle $= \frac{1}{2}RC$ and $C = 2\pi R$.

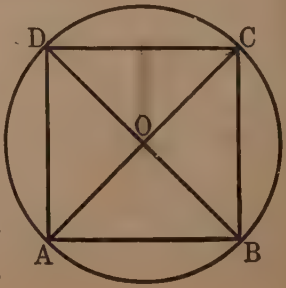
$\therefore$ By substitution, the area of a circle $= \frac{1}{2}R(2\pi R)$ or $\mathbf{A} = \pi\mathbf{R}^2$.

471. **Corollary 2.**—The areas of two circles are to each other as the squares of their radii. For if A and A' denote the areas, and R and R' the radii, then $A:A' = \pi R^2:\pi R'^2$ or $A:A' = R^2:R'^2$.

472. **Corollary 3.**—The area of a sector is equal to one-half the product of its radius and bounding arc.

473. **Corollary 4.**—Similar sectors are to each other as the squares of their radii. **Hint.**—*Similar sectors are sectors whose central angles are equal.*

474. **Problem.**—To inscribe a square in a given circle.



Given the circle whose center is O.

To inscribe a square in the circle O.

Construction.—Draw the diameters AC and $BD \perp$ to each other.

Draw AB, BC, CD, and DA.

$ABCD$ is the required square.

Proof. $\angle$s A, B, C, and D are rt. $\angle$s. Why?

$AB = BC = CD = DA$. Why?

$\therefore ABCD$ is a square. Q.E.F.

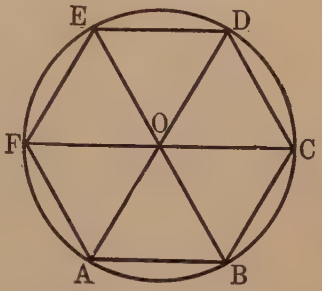
3.14

PROBLEMS

1. Inscribe a regular octagon in a circle; regular polygons of 16, 32, 64 sides respectively.

2. Show that by drawing tangents to a circle at the vertices of an inscribed square, a circumscribed square is formed whose area is twice that of the inscribed square.

3. Draw two concentric circles and show that the area of the circular ring is equal to that of a circle whose diameter is a chord of the outer circle and a tangent to the inner circle.



475. Problem.—To inscribe a regular hexagon in a circle.

Given the circle whose center is *O*.

To inscribe a regular hexagon in the circle *O*.

Construction.—Draw the radius *OA*. Draw a chord *AB* = the radius *OA*. Then chord *AB* is one side of the regular hexagon.

Proof.—Draw *OB* a radius, and prove that $\angle AOB = 60°$ or $\frac{1}{6}$ of 360°.

∴ The arc *AB* is $\frac{1}{6}$ of a circumference and *AB* is one side of a regular inscribed hexagon. *Q.E.F.*

PROBLEMS

1. Find the ratio of the areas of an inscribed and a circumscribed square.

2. Inscribe a regular twelve sided figure in a circle.

3. Inscribe an equilateral triangle in a circle.

4. Circumscribe an equilateral triangle about the circle in problem 3.

5. Find the ratio of the sides of the triangles in problems 3 and 4.

6. Compare the areas of the triangles in problems 3 and 4.

7. Compute the side, apothem, and area of a square inscribed in a circle whose radius is 10 feet.

8. Compute the side, apothem, and area of a square circumscribed about a circle whose radius is 20 inches.

9. An equilateral triangle is inscribed in a circle whose radius is 10 inches. Find a side, an altitude, and the area.

10. Compare the area of the triangle in problem 9 with that of the circle.

11. An equilateral triangle is circumscribed about the circle in problem 9. Find the area and compare it with area found in problem 9.

12. Compute the radius of a circle circumscribed about an equilateral triangle whose side is 20 inches.

13. Find the area of the circle in problem 12.

14. A regular hexagon is inscribed in a circle with a radius of 8 feet. Find the apothem and area.

15. Find the radius of a circle circumscribed about a hexagon whose perimeter is 60 feet.

16. Find the area of a circle whose diameter is 100 feet.

17. Compute the number of degrees in the interior angles of each of the following regular polygons; pentagon: hexagon; octagon; decagon.

18. Compute the number of degrees in an angle at the center of each of the above polygons.

19. How many sides has a regular polygon each of whose interior angles equals: (*a*) 120°; (*b*) 135°; (*c*) 144°; (*d*) $128\frac{4}{7}$°; (*e*) 140°?

20. Find the number of degrees in a central angle of each of the polygons of problem 19.

21. Find the difference in area between a circle and its inscribed hexagon, if the radius of the circle is 10 feet.

22. The area of a circle is 78.54 sq. in. Find the radius.

23. The circumference of a circle is 87.9648. Find the radius.

24. A circular plot is to contain one acre (43,560 sq. ft.). Find the radius.

25. The radius of a circle is 10 inches. Find the radius of a circle twice as large; one $\frac{3}{4}$ as large.

26. Find the area of a sector whose central angle is 40° and whose radius is 14 inches.

27. Find the area of a circular segment which subtends a central angle of 60°, and whose radius is 12 inches.

28. The radii of two circles are 5 and 10. Find the ratio of their areas.

29. Compare the area of a circle with that of a triangle whose base equals the circumference of the circle and altitude equals the radius.

30. How many revolutions will a wheel 34 inches in diameter make per mile?

31. There are 250 tubes each 3 inches in diameter in the boiler of a heating plant. Find the total area of their cross sections.

32. An oil tank is 15 feet in diameter. Find the total length of metal required to make four straps around the tank, allowing 6 inches for overlapping for each strap.

33. Two pulleys are 12 feet apart, center to center, and each has a radius of 7 inches. Find the length of belting required to engage these pulleys.

34. A running track has two parallel sides $\frac{1}{16}$ of a mile long and semicircular ends each $\frac{1}{16}$ of a mile in length on the inner side. What actual distance is covered by a runner whose path is 12 inches from the inner side?

35. What pressure is exerted on a piston head in a cylinder of 18 inches radius, if the steam pressure in the boiler is 80 lbs. per square inch?

36. A water main feeds a series of 12 half-inch pipes. What must be its diameter to supply these pipes with a constant flow of water?

37. A hot air furnace draws cold air from the outside to supply 8 pipes each 10 inches in diameter which carry the heated air into the building. What must be the minimum size of the cross section of the cold air duct, not allowing for expansion of the heated air?

38. The side of a regular inscribed hexagon is 20 inches. Find the radius, apothem, and area of the hexagon.

39. Find the area of the regular hexagon circumscribed about the circle of the above problem.

40. Find the area of the equilateral triangle inscribed in the above circle.

41. The radii of two concentric circles are R and r. Develop a formula for finding the area of the circular ring thus formed. Denote this area by A.

42. The circumference of the top of a smoke stack is one-half as large as the circumference at the base, compare their radii.

43. The radii of a driving wheel of a locomotive and a small trailer wheel are as 4:1. Compare the number of revolutions made by these wheels in going any given distance.

44. A chord 50 inches long subtends an arc 20 inches high. Find the radius of the circle.

45. An arc 5 feet long in a circle whose radius is 14 feet subtends how many degrees at the center? (Use $\pi = \frac{22}{7}$.)

46. If a 4 inch pipe fills a tank in three hours, how long will it take a $1\frac{1}{2}$ inch pipe?

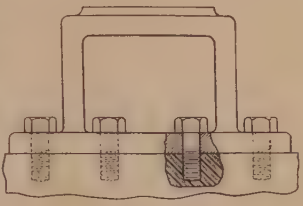
47. Find the radius of a circle equal in area to the sum of the areas of two circles whose radii are 8 and 10 inches.

48. Prove that the side of a circumscribed regular hexagon is $\frac{2}{3}R\sqrt{3}$, where the radius is denoted by R.

49. Find the side of an equilateral triangle circumscribed about a circle whose radius is R.

50. An iron casting is bolted to the floor by four $1\frac{1}{4}''$ diameter mild steel bolts, and a force tends to slide the casting along the floor. Neglecting friction, what is the force when the unit shearing stress in the bolts is 10,000 pounds per square inch?

Force = total area of a section × stress.



Prob. 50.

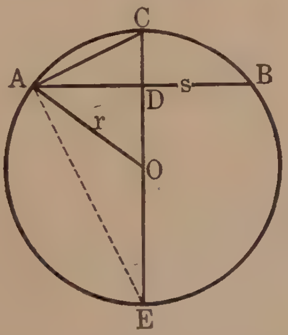
51. A short square steel bar $2'' \times 2''$ in section and $4'$ long carries a tensile load 60,000 pounds. The Modulus of Elasticity is 30,000,000 lbs. per square inch. What is the unit tensile strength and total elongation?

$$\text{Stress} = \frac{\text{load}}{\text{area}}$$

$$\text{elongation} = \frac{\text{load} \times \text{length in ins.}}{\text{area} \times \text{mod. of El.}}$$

Prob. 51.

476. Theorem.—If s is the side of a regular polygon inscribed in a circle of radius r, then the side of a regular inscribed polygon of double the number of sides is $\sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.



Given the side AB of a regular inscribed polygon denoted by s, the radius denoted by r, and AC the side of a regular inscribed polygon of double the number of sides.

To prove that $AC = \sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.

Proof.—Draw the diameter CE cutting AB at D. Draw AO and AE.

Now $CE \perp AB$ at its middle point. §169.

In rt. $\triangle$ DAO we have, $\overline{OD}^2 = r^2 - \left(\frac{s}{2}\right)^2$. §412.

$$\therefore OD = \sqrt{r^2 - \frac{s^2}{4}}.$$

Now $CD = r - OD$, or $r - \sqrt{r^2 - \frac{s^2}{4}}$.

Since CAE is a right triangle $\overline{AC}^2 = CD \times CE$. §405–3.

$\therefore AC = \sqrt{CD \times CE}$.

Substituting, $AC = \sqrt{\left(r - \sqrt{r^2 - \frac{s^2}{4}}\right)(2r)}.$

$$\therefore\ AC = \sqrt{2r^2 - 2r\sqrt{r^2 - \frac{s^2}{4}}}.$$

$$\therefore\ AC = \sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$$

Q.E.D.

477. **To Compute the Value of π Approximately.**—If a regular hexagon is inscribed in a circle whose radius is 1, then the side of the hexagon is 1 and the perimeter is 6. If in this same circle we double the number of sides of the hexagon seven successive times we will get regular polygons of 12, 24, 48, 96, 192, 384, and 768 sides.

By using the formula, $AC = \sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$, just developed for finding the side of a regular polygon of double the number of sides, we can find the length of the side of each succeeding polygon. The perimeter is found by multiplying a side of each polygon by the number of sides of that particular polygon. Thus,

Number of sides of polygon	Length of side	Perimeter
6	1	6
12	$\sqrt{2 - \sqrt{4 - (1)^2}} = 0.51763809$	6.21165708
24	$\sqrt{2 - \sqrt{4 - (.51763809)^2}} = 0.26105238$	6.26525722
768	$\sqrt{2 - \sqrt{4 - (.01636228)^2}} = 0.00818121$	6.28316941

But consider a polygon of 768 sides inscribed in a circle whose radius is 1 inch, for example, and remember that we have computed its perimeter to be 6.28316941 inches and consider that for all practical purposes of measurement this perimeter of 768 sides is approximately equal to the circumference of the circle, then we see that **C** equals 6.28316941. (Nearly.)

But $C = 2\pi r$ or $\pi = \frac{C}{2r}$. §466.

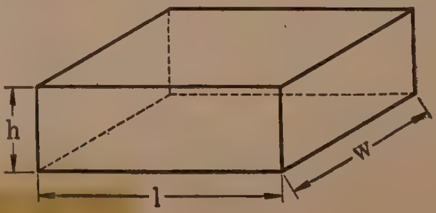
Substituting, $\therefore\ \pi = \frac{6.28316941}{2 \times 1}$ or 3.14158+

The value of π used depends upon the accuracy required.
For most purposes $\pi = 3.1416$.

MENSURATION OF SOLIDS

478. **Rectangular Solid.**—A rectangular solid has six rectangular faces as shown. The lateral faces are the sides and the end faces are the bases.

479. **Lateral Edge.**—The lateral edge of a solid is the line of intersection of two lateral faces.

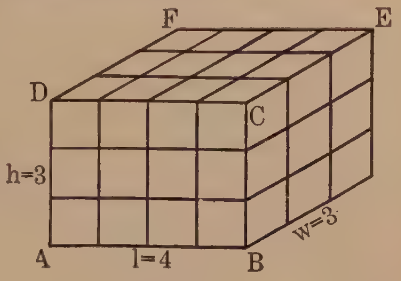


480. **Lateral Area.**—The lateral area of a solid is the sum of the areas of the lateral faces.

481. **Total Area.**—The total area of a solid is the sum of the lateral area and the area of its bases.

482. **Formula for the Volume of a Rectangular Solid.**

$$V = lwh$$



Think of a rectangular solid, 4 by 3 by 3 inches, as divided into blocks of one cubic inch each.

It contains 36 "cubic-inch" blocks having 3 tiers of 12 each. This value is the product of length, width, and height.

$\therefore V = 4 \times 3 \times 3$ or 36 cu. in.

PROBLEMS

Find the volume of a rectangular solid whose dimensions are:

1. 8, 9, and 10.
2. 10, 13, and 15.
3. 6.5, 8.6, and 9.4.
4. 10.7, 8.92, and 31.6.
5. 5.17, 8.73, and 9.15.
6. 10.07, 9.19, and 32.06.

7. A fill of uniform depth of 3.5 ft., 30 ft. wide, 500 ft. long is required. How many loads of 1 cu. yd. each must be moved?

8. How many cu. ft. of earth must be removed in making an excavation 30′ × 50′ × 6′?

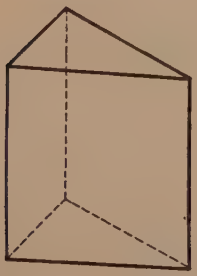
9. How many cu. yds. is contained in a car 40.5′ × 8.5′ × 4′?

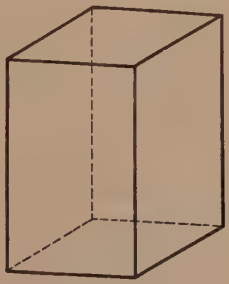
10. A coal pocket 40 ft. long, 10 ft. wide, and 20 ft. high is $\frac{3}{5}$ full. How high will the coal stand if leveled and how many tons will it contain, figuring 35 cu. ft. to the ton?

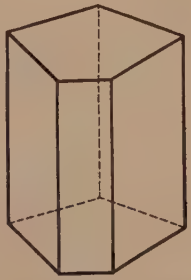
11. A rectangular tank 20.5′ × 30.2′ × 5.7′ is filled with oil. How many gallons will it contain allowing $7\frac{1}{2}$ gallons to the cu. ft.?

12. How many cu. ft. of air is contained in a room 30′ 6″ × 20′ 8″ × 12′?

483. **Prism.**—A prism is a solid whose lateral faces are parallelograms and whose bases are equal and parallel polygons.


Triangular Prism


Rectangular Prism


Pentagonal Prism

484. **Altitude.**—The altitude of a prism is the perpendicular distance between the planes of its bases.

485. Formula for the Volume of Any Prism.

$$V = Bh,$$

where **B** is the area of the base and **h** is the altitude.

PROBLEMS

Find the volume of the following prisms.

1. Base = 40 sq. in., altitude = 20 in.
2. Base = 18.8 sq. in., altitude = 9.2 in.
3. Base = 20.52 sq. in., altitude = 5.18 in.
4. Find the volume of a rectangular prism whose base is a rectangle 7″ × 9″ and whose altitude is 15″.
5. Find the altitude of a prism whose volume is 300 cu. ft. and whose base is 15 sq. ft.

486. **Right Prism.**—A right prism is a prism whose lateral edges are perpendicular to the bases.

487. Formula for the Lateral Area of a Right Prism.

$$\text{L.A.} = ph$$

where **L.A.** is the lateral area, **p** is the perimeter of the base, and **h** is the altitude.

488. Formula for the Total Area of a Right Prism.

$$\text{T.A.} = \text{L.A.} + 2B,$$

where **T.A.** is the total area and **B** is the base area.

PROBLEMS

1. Find the lateral area of a right prism whose altitude is 22 inches and the perimeter of whose base is 32 inches.
2. Find the lateral area and total area of a right prism having an altitude of 12.5 inches and a triangular base whose sides are 8, 9, and 15 inches.
3. Find the total area of a right prism whose altitude is 25 inches and a regular hexagonal base, each of whose sides is 6 inches.
4. Find the total area of a right prism the sides of whose triangular base are 20, 22, and 28 and whose altitude is 40.

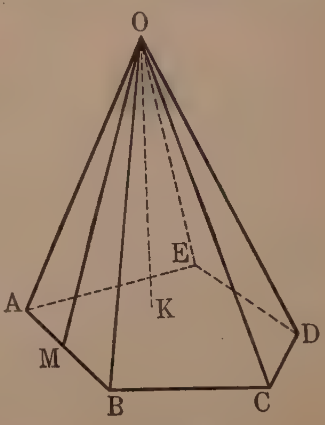
489. **Pyramid.**—A pyramid is a solid whose base is a polygon and whose lateral faces are triangles having a common vertex.

The sum of the areas of its lateral faces is the **lateral area.**

The intersections of its lateral faces are **lateral edges.**

The **altitude** is the perpendicular distance from the vertex to the plane of the base. A **regular pyramid** is a pyramid whose base is a regular polygon and whose center coincides with the foot of the perpendicular dropped from the vertex to the base.

The **slant height** of a regular pyramid is the altitude of one of its triangular faces.



Pyramid

OK is the altitude,
OM is the slant height,
OA, *OB*, *OC*, *OD*, and *OE* are the lateral edges.

490. **Formula for the Volume of a Pyramid.**

$$V = \tfrac{1}{3}Bh.$$

491. **Formula for the Lateral Area of a Regular Pyramid.**

$$L.A. = \tfrac{1}{2}sp,$$

where **s** is the slant height.

492. **Formula for the Total Area of a Regular Pyramid.**

$$T.A = L.A + B.$$

PROBLEMS

Find the volume of a regular pyramid:

1. When the altitude is 12 inches and the base is a triangle whose sides are each 8 inches.

2. When the base is a square each of whose sides is 5.5 feet and the altitude is 9.7 feet.

3. When the altitude is 9 feet and each side of the hexagonal base is 3.5 feet.

4. Find the total area of a pyramid, if each side of the triangular base is 4 inches and the slant height is 7.8 inches.

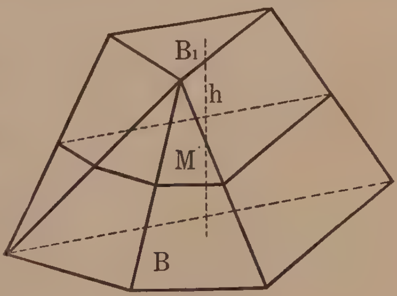
5. Find the total area of a pyramid, if each side of the square base is 5.3 feet and the slant height is 7.2 feet.

6. Find the lateral surface of a pyramid if each side of its pentagonal base is 5.27′ and the slant height is 19.18′.

7. The volume of a triangular pyramid is 24 cu. ft. The sides of the base are 4, 5, and 7 ft. Find the height.

493. **Prismatoid.**—A prismatoid is a solid two of whose faces (the bases) are polygons in parallel planes.

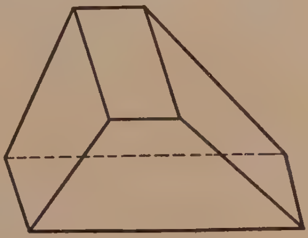
The sides are triangles or trapezoids, having one side common with a side of one base and the opposite vertex or side common with a vertex or side of the other base. **The altitude** is the perpendicular distance between the bases. The mid-section is a plane cutting all the edges midway between and parallel to the bases. It bisects all the edges and the altitude.



Prismatoid

In the above figure **h** is the altitude, **B** and **B₁**, the areas of the lower and upper bases respectively, and **M**, the area of the mid-section.

494. **The Prismatoid as a General Solid.**—(*a*) A prismatoid in which all faces are plane figures, and the two ends are **dissimilar** parallel plane figures of the same number of sides is **a prismoid.**



Prismoid

(*b*) When the bases of a prismatoid are equal polygons and the corresponding sides are parallel, the figure is **a prism.**

(*c*) When the bases are unequal but similar, and the corresponding sides parallel, a **frustum** of a **pyramid** is formed.

(*d*) When one base is a point the figure is a **pyramid.**

(*e*) When one base is a line the figure is a **wedge.**

495. **Formula for the Volume of a Prismatoid.**

$$V = \tfrac{1}{6}h(B + B_1 + 4M).$$

Engineers use this formula in computing the volume of earth removed in making cuts or fills in railway construction.

PROBLEMS

Solve the following by the prismatoid formula.

1. Find the volume of a prism whose base is a square 10 inches on a side, and whose altitude is 15 inches.

2. Find the volume of a pyramid whose base is a regular hexagon 6 inches on a side, and whose altitude is 21 inches.

3. Find the volume of earth to be moved in excavating a trench 100 feet long with sloping sides. A cross section of the trench is 30 feet wide at the top, 15 feet wide at the bottom, and has a depth of 9 feet.

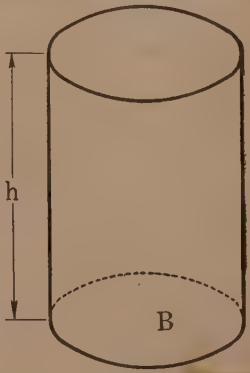
4. Find the volume of a rectangular wedge having a base 80 ft. by 50 ft., whose upper edge is 70 ft., and altitude 50 ft.

5. Find the number of cubic yards of earth removed in making a cut 200′ long, if one end is 18′ deep, 75′ wide at the top, and 24′ wide at the bottom; the other end is 14′ deep, 66′ wide at the top, and 24′ wide at the bottom.

496. **Cylindrical Surface.**—A cylindrical surface is a curved surface generated by a moving straight line which always remains parallel to its first position and which always touches a given curved line not in the plane of the straight line. The moving line in any position is an **element** of the cylindrical surface.

497. **Cylinder.**—A cylinder is a solid bounded by two parallel planes and a cylindrical surface.

The parallel planes are the **bases.** The cylindrical surface is the **lateral surface.** The **altitude** is the perpendicular distance between the bases. If the elements are perpendicular to the base the cylinder is a **right cylinder.** If the bases are circles it is a **circular cylinder.** The area of its lateral surface is the **lateral area.** The lateral area plus the two base areas is the **total area.**



Cylinder

498. **Formula for the Volume of a Cylinder.**

$$V = Bh.$$

499. **Formula for the Lateral Area of a Right Circular Cylinder.**

$$L.A. = Ch.$$

500. Formula for the Total Area of a Right Circular Cylinder.

$$T.A. = L.A. + 2B.$$

2cπ

PROBLEMS

1. The diameter of a cylindrical tank is 8 feet and the height is 12 feet. Compute its volume. How many gallons of water will it hold if there are 7½ gallons in a cubic foot?

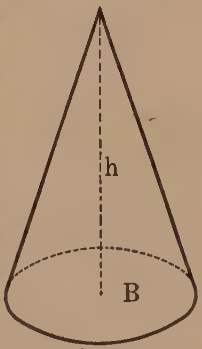
2. How many square feet of sheet iron is required to make a cylinder with open ends, if it is 20 feet long and 20 inches in diameter?

3. Find the length of a cylindrical tank 2.5 feet in diameter which will hold 85 gallons.

4. How many cu. yds. of earth must be removed in digging a tunnel 30 ft. long, whose cross section is a semi-circle with a radius of 28 ft.?

501. Conical Surface.—A conical surface is a surface generated by a moving straight line which constantly touches a fixed curve and passes through a fixed point not in the plane of the curve. The moving line in any position is an **element** of the conical surface.

502. Cone.—A cone is a solid bounded by a conical surface and a plane cutting this surface. The conical surface is the **lateral surface** of the cone. The plane surface is the **base.** The fixed point is the **vertex.** The perpendicular distance from the vertex to the plane of the base is the **altitude.** A **circular cone** is one whose base is a circle. A line drawn from the vertex to the center of the base is the **axis.** If the base is a circle and the axis is perpendicular to the base, the cone is a **right circular cone.** The **slant height** of a right circular cone is an element of the cone.



Cone

503. Formula for the Volume of a Circular Cone.

$$V = \tfrac{1}{3}Bh.$$

504. Formula for the Lateral Area of a Right Cone.

$$L.A. = \tfrac{1}{2}Cs,$$

where **s** is an element.

505. Formula for the Total Area of a Right Cone.

$$T.A. = L.A. + B.$$

PROBLEMS

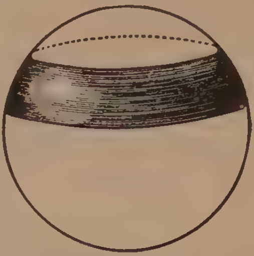
1. Find the volume of a right circular cone having an altitude of 10 feet, if the radius of the base is 1.5 feet.

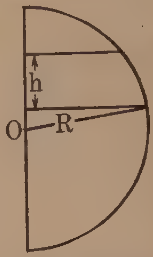
2. Find the lateral and total areas of a right cone whose slant height is 12.7 feet, and the circumference of the base is 16.2 feet.

3. The height of a right circular cone is 8 feet. The diameter of the base is also 8 feet. Find the ratio of the area of the base to the lateral area.

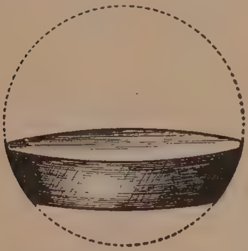
506. Sphere.—A sphere is a solid bounded by a surface all points of which are equally distant from a point within called the center. The **radius** of a sphere is a straight line drawn from the center to any point in the surface. The **diameter** of a sphere is a line passing through the center and terminated by the surface. The **area** of a sphere is the area of its surface.

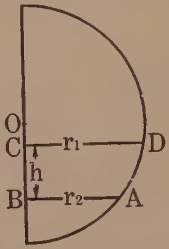
507. **Zone.**—A **zone** of a sphere is that portion of its surface included between two parallel planes which cut its surface. The circumferences made by the cutting planes are the **bases** of the zone and the perpendicular distance between those planes is the **altitude** of the zone. If one base is tangent to the sphere, the zone is a **zone of one base.**

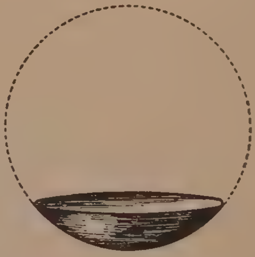




508. **Segment of a Sphere.**—A segment of a sphere is that portion of a sphere included between two parallel planes which cut the surface of the sphere. The parallel planes are its **bases** and the perpendicular distance between its bases is its **altitude.** If one plane is tangent to the sphere the segment is a **segment of one base.**







509. **Formula for the Area of the Surface of a Sphere.**

$$A = 4\pi R^2,$$

where **A** is the area and **R** the radius.

510. **Formula for the Area of a Zone.**

$$A = 2\pi Rh,$$

where **A** is the area, **R** a radius of the sphere, and **h** the altitude.

511. **Formula for the Volume of a Sphere.**

$$V = \tfrac{1}{3}RA \text{ or } \tfrac{4}{3}\pi R^3 \text{ or } \tfrac{1}{6}\pi D^3.$$

512. **Formula for the Volume of a Segment of Two Bases.**

$$V = \frac{h}{2}(\pi r_1^2 + \pi r_2^2) + \frac{\pi h^3}{6},$$

where r_1 is the radius of the circular cross section of one base and r_2 is the radius of the circular cross section of the other base. If the spherical segment has but one base, r_2 becomes zero and the formula reduces to,

$$V = \tfrac{1}{2}\pi r^2 h + \tfrac{1}{6}\pi h^3.$$

Note.—*The prismatoid formula is general and can be used for finding the volume of any solid.*

PROBLEMS

1. Find the surface of a sphere if the diameter is: (*a*) 12′; (*b*) 13.5′; (*c*) 28.3′.

2. Find the volume of a sphere if the diameter is: (*a*) 8.2′; (*b*) 9.3′; (*c*) 18.7′; (*d*) 15.8′.

3. A dome in the shape of a half-sphere is 60 feet in diameter. How many sq. ft. of metal is required to cover it?

4. A ball 4.5 feet in diameter is gilded at 50 cents per sq. ft. Compute the cost.

5. The volume of a sphere is 33.5104 cu. ft. Find the diameter.

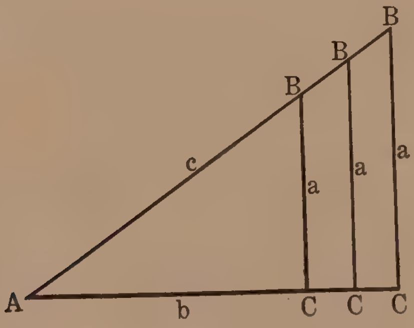
6. The surface of a sphere is 50.2656 sq. ft. Find the diameter.

CHAPTER XIII

RELATIONS OF THE FUNCTIONS OF ANGLES

513. **Functions of Angles.**—When two variables, such as x and y, are so related that a change in the value of $\mathbf{x}$ causes a corresponding change in the value of $\mathbf{y}$, then $\mathbf{y}$ is said to be a function of $\mathbf{x}$ (§267). By referring to the table for the sine, the cosine, and the tangent of various angles (§289) it is seen that as the angle changes there is also a corresponding change in the value of the sine, the cosine, and the tangent of those angles. For this reason these ratios are called the functions of the angle.

They should be remembered as the ratios of the sides of a right triangle as follows:



$$\text{sin of acute } \angle = \frac{\text{opp. side}}{\text{hyp.}}. \qquad \text{cot of acute } \angle = \frac{\text{adj. side}}{\text{opp. side}}.$$

$$\text{cos of acute } \angle = \frac{\text{adj. side}}{\text{hyp.}}. \qquad \text{sec of acute } \angle = \frac{\text{hyp.}}{\text{adj. side}}.$$

$$\text{tan of acute } \angle = \frac{\text{opp. side}}{\text{adj. side}}. \qquad \text{csc of acute } \angle = \frac{\text{hyp.}}{\text{opp. side}}.$$

The ratios which define the functions are true regardless of the length of the sides of the triangle so long as the acute angle does not change in value. This is readily understood from the figure, since all the triangles are similar and the ratios of the corresponding sides are equal.

Three other functions are sometimes used in engineering, viz., the ex-secant (exsec), the versed sine (vers), and the coversed sine (covers). They are defined as follows:

$\mathbf{exsec\ A = sec\ A - 1}$, **vers** $\mathbf{A = 1 - cos\ A}$, **and covers** $\mathbf{A = 1 - sin\ A}$.

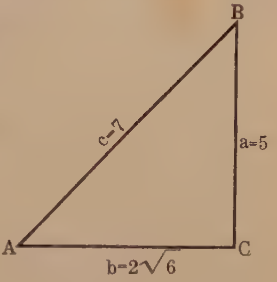
PROBLEMS

1. Write the six functions of A and B in the right triangle ABC where $\angle C$ is the right angle.

2. From problem 1 show that the function of an angle is equal to the co-function of its complement.

Thus, $\tan A = \frac{a}{b} = \cot B$.

3. In the right triangle ABC, $a = 5$ and $b = 2\sqrt{6}$, find the six functions of A and B.



Solution. $a^2 + b^2 = c^2$.

Substituting, $25 + 24 = c^2$.

$\therefore c = 7$.

Writing the functions from the figure we have,

$$\sin A = \frac{5}{7} = \cos B. \qquad \cot A = \frac{2\sqrt{6}}{5} = \tan B.$$

$$\cos A = \frac{2\sqrt{6}}{7} = \sin B. \qquad \sec A = \frac{7}{2\sqrt{6}} = \frac{7}{12}\sqrt{6} = \csc B.$$

$$\tan A = \frac{5}{2\sqrt{6}} = \frac{5}{12}\sqrt{6} = \cot B. \qquad \csc A = \frac{7}{5} = \sec B.$$

4. What condition must be fulfilled so that the three sides a, b, and c will be the sides of a right triangle? Is this condition fulfilled in the following values of a, b, and c? When it is, write the six functions of angle A.

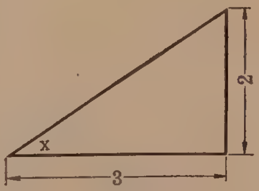
(*a*) 3, 4, and 5. (*b*) 8, 15, and 17.
(*c*) 5, 12, and 13. (*d*) 7, 10, and 12.
(*e*) 1, $\sqrt{3}$, and 2. (*f*) $2ab$, $a^2 - b^2$, and $a^2 + b^2$.
(*g*) 1.20, 1.19, and 1.69. (*h*) $\frac{2mn}{m-n}$, $m + n$, and $\frac{m^2 + n^2}{m - n}$.
(*i*) $\frac{2}{3}$, $\frac{1}{3}$, and $\frac{\sqrt{5}}{3}$.

5. If $a^2 + b^2 = c^2$, find the functions of A in the following: (Draw figures.)

(*a*) $a = 6$, $b = 8$. (*b*) $a = 4$, $b = 6$. (*c*) $a = .3$, $c = .5$.
(*d*) $b = 12$, $c = 13$. (*e*) $a = x^2 - y^2$, $b = 2xy$. (*f*) $b = 3a$, $c = 5a$.
(*g*) $a = \sqrt{3}$, $c = 2$. (*h*) $a = 2$, $b = \sqrt{5}$.

6. On coordinate paper construct x if,

(*a*) $\tan x = \frac{2}{3}$. (*b*) $\sin x = \frac{4}{5}$. (*c*) $\cot x = \frac{5}{3}$.
(*d*) $\sec x = \sqrt{2}$. (*e*) $\csc x = \frac{8}{5}$. (*f*) $\cos x = \frac{5}{6}$.
(*g*) $\sec x = \frac{8}{3}$. (*h*) $\tan x = \sqrt{3}$.

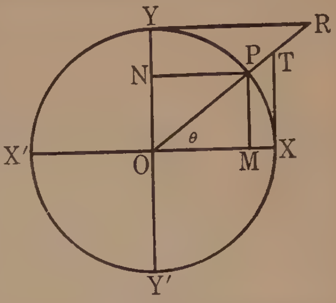


PROB. 6*a*.

7. Draw a right triangle and solve the following:

(*a*) If $a = \frac{1}{3}c$, find the six functions of A.
(*b*) If $b = \frac{1}{2}c$, find the six functions of B.
(*c*) If $a = 3b$, find the six functions of A.
(*d*) If $c = 2a$, find the six functions of B.
(*e*) If $b = 2.64$ and $c = 2.65$, find the six functions of A.
(*f*) Find **a**, if $\sin A = \frac{4}{5}$, and $c = 25.2$.
(*g*) Find **b**, if $\tan A = \frac{1}{52}$ and $a = 3.5$.
(*h*) Find **c**, if $\csc A = 5.3$ and $a = 23$.
(*i*) Find **c**, if $\cos A = .45$, and $b = 4.35$.
(*j*) Find **b**, if $\cot A = 1.3$ and $a = 5.2$.
(*k*) If $c = 3.53$ feet, $\sin A = 0.8$, and $\cos A = 0.6$; find the legs.

514. Line Representation of the Functions of an Angle.—Let O be the center of a unit circle. (A unit circle is one in which the radius is unity or 1.) Draw the diameters XX' and YY' perpendicular to each other. Draw OR making any convenient angle, θ (Theta) with OX. Draw $PM \perp OX, PN \perp OY, TX$ tangent at X, and YR tangent at Y.



Note that $\angle\ YRO = \angle\ \theta$, (Alternate interior $\angle$s of $\|$ lines, etc.) and keep in mind that the radius of the circle is 1.

Since OP, OX, and OY are radii and equal to 1, the functions may be defined and represented as a single line, as follows:

$$\sin\theta = \left(\frac{PM}{OP}\right) = \text{PM}. \qquad \cot\theta = \left(\frac{YR}{OY}\right) = \text{YR}.$$

$$\cos\theta = \left(\frac{PN}{OP}\right) = \left(\frac{OM}{OP}\right) = \text{OM}. \qquad \sec\theta = \left(\frac{OT}{OX}\right) = \text{OT}.$$

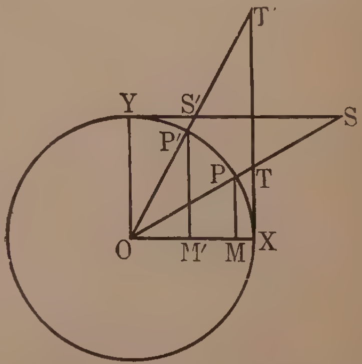
$$\tan\theta = \left(\frac{TX}{OX}\right) = \text{TX}. \qquad \csc\theta = \left(\frac{OR}{OY}\right) = \text{OR}.$$

$$\text{exsec}\ \theta = \text{PT}. \qquad \text{vers}\ \theta = \text{MX}. \qquad \text{covers}\ \theta = \text{YN}.$$

515. Infinity.—The sign ∞ is used to denote infinity which is larger than any assignable value. It is **not** a number.

Some idea of the meaning of infinity may be gained if we consider the fraction $\frac{3}{x}$, in which the denominator **x** decreases indefinitely, becoming for example, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc., until it approaches zero. In each case the value of the quotient becomes correspondingly larger and approaches a value larger than any known number. This result is called **infinity.**

516. Change in the Values of the Functions as the Angle Increases from 0° to 90°.—If the angle θ is increased from 0° to 90°, the values of the functions will either increase or decrease. The lines $PM, P'M'$, etc., represent $\sin\theta$; OM, OM', etc. $\cos\theta$; XT, XT', etc., $\tan\theta$; YS, YS', etc., $\cot\theta$; OT, OT', etc., the sec θ; and OS, OS', etc., the csc θ.



Note.—*The sine, tangent, and secant* **increase** *as the angle increases; while the cosine, cotangent, and cosecant* **decrease** *as the angle increases.*

The reverse is also true if the angle decreases in value.

By a careful study of the figure it will be seen that as the angle θ increases from 0° to 90°, $\sin\theta$ increases from 0 to 1; $\cos\theta$ decreases from

1 to 0; tan θ increases from 0 to ∞, cot θ decreases from ∞ to 0; sec θ increases from 1 to ∞; and csc θ decreases from ∞ to 1.

These are the limiting values of the functions of an acute angle.

Hence, for acute angles the sines and cosines can have values between 0 and 1 only; tangents and cotangents can have any value; and secants and cosecants can have any value greater than 1, but can never have a value between 0 and 1.

§513 can now be more fully appreciated.

PROBLEMS

1. If θ is an acute angle, show that:

(*a*) tan θ is greater than sin θ. (*b*) tan θ is less than sec θ. (*c*) cot θ is less than csc θ. (*d*) sec θ is greater than cos θ.

2. Find θ, if sin θ is represented by one-half the side of a regular inscribed hexagon; one-half the side of a regular inscribed octagon. §456.

3. Construct the angle θ, if:

(*a*) $\sin \theta = 3 \cos \theta$. (*c*) $\sec \theta = 3$. (*e*) $3 \sin \theta = \tan \theta$.
(*b*) $\tan \theta = 2$. (*d*) $\cos \theta = \sin \theta$. (*f*) $\cos \theta = \frac{2}{3}$.

4. Given an angle A; construct an angle B such that:

(*a*) $\sin B = 3 \sin A$. (*c*) $\csc B = \sec A$.
(*b*) $\tan B = 2 \tan A$. (*d*) $2 \cos B = \cos A$.

5. Show that twice the sine of an angle is equal to the chord of twice the angle.

517. Functions of Complementary Angles.—As has been shown in §289 the function of an angle is the co-function of its complement. In fact that is where the co-functions get their name. Cosine means complements' sine.

(*a*) $\sin A = \cos B = \cos (90° - A)$.
(*b*) $\cos A = \sin B = \sin (90° - A)$.
(*c*) $\tan A = \cot B = \cot (90° - A)$.
(*d*) $\cot A = \tan B = \tan (90° - A)$.
(*e*) $\sec A = \csc B = \csc (90° - A)$.
(*f*) $\csc A = \sec B = \sec (90° - A)$.

PROBLEMS

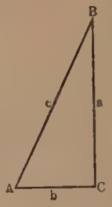
1. Express the following functions as a function of the complementary angle.

(*a*) sin 40°. (*b*) cos 45°. (*c*) tan 58°.
(*d*) cot 42°. (*e*) sec 30°. (*f*) csc 48°.
(*g*) sin 32° 25′. (*h*) tan 47° 53′. (*i*) sec 83° 59′.

2. Given $\sin 30° = \frac{1}{2}$, find cos 60°.
3. Given $\tan 60° = \sqrt{3}$, find cot 30°.
4. Given $\sin \theta = \cos \theta$, find θ.
5. Given $\tan \theta = \cot 2\theta$, find θ.
6. Given $\sec 3\theta = \csc 6\theta$, find θ.
7. Given $\tan \theta = \cot (45° - \frac{1}{2}\theta)$, find θ.
8. Given $\sin \frac{1}{2}A = \cos 2A$, find A.
9. Given $\cot 3A = \tan 12A$, find A.
10. Given $\csc (45° + x) = \sec x$, find x.
11. Given $\sin x = \cos mx$, find x.
12. Given $\tan (38° + x) = \cot (x - 40°)$, find x.

518. Formulas Connecting the Functions of an Angle.

The sum of the squares of the sine and cosine of an angle equals 1.



From the triangle ABC, $a^2 + b^2 = c^2$.

Dividing by c^2, $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$.

But $\frac{a^2}{c^2} = \sin^2 A$ and $\frac{b^2}{c^2} = \cos^2 A$.

Substituting, $\sin^2 A + \cos^2 A = 1$. (1)

Solving (1) for $\sin A$ and $\cos A$ we get, $\sin A = \sqrt{1 - \cos^2 A}$ and $\cos A = \sqrt{1 - \sin^2 A}$.

The tangent of an angle is equal to the sine divided by the cosine. This can be shown to be true from triangle ABC.

Thus, $\tan A = \frac{a}{b}$, $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$.

$$\text{Hence } \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b}.$$

$$\therefore \tan A = \frac{\sin A.}{\cos A.} \qquad (2)$$

Show from the above figure that:

(*a*) *The sine and cosecant of an angle are reciprocals.*

$$\sin A = \frac{1}{\csc A}, \quad \csc A = \frac{1}{\sin A}, \quad \text{and } \sin A \csc A = 1. \qquad (3)$$

(*b*) *The cosine and secant of an angle are reciprocals.*

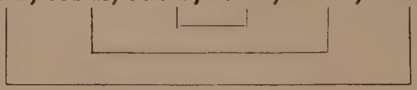
$$\cos A = \frac{1}{\sec A}, \quad \sec A = \frac{1}{\cos A}, \quad \text{and } \cos A \sec A = 1. \qquad (4)$$

(*c*) *The tangent and cotangent of an angle are reciprocals.*

$$\tan A = \frac{1}{\cot A}, \quad \cot A = \frac{1}{\tan A}, \quad \text{and } \tan A \cot A = 1. \qquad (5)$$

The following arrangement will aid in remembering the reciprocal relations between the functions.

sin A, cos A, tan A, cot A, sec A, csc A.



(*d*) $\tan^2 A = \sec^2 A - 1$, and $\sec^2 A = 1 + \tan^2 A$. (6)

(*c*) $\cot^2 A = \csc^2 A - 1$, and $\csc^2 A = 1 + \cot^2 A$. (7)

It can be shown that the above formulas are true for angles of any magnitude.

By the use of the above formulas, any function of an angle can be derived in terms of any other function.

PROBLEMS

Show that the following identities are true.

1. $\sec^2 A \csc^2 A = \sec^2 A + \csc^2 A$.

Solution. $\dfrac{1}{\sin^2 A \cos^2 A} = \dfrac{1}{\cos^2 A} + \dfrac{1}{\sin^2 A} = \dfrac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A}$.

But, $\sin^2 A + \cos^2 A = 1$. Why?

Hence, $\dfrac{1}{\sin^2 A \cos^2 A} \equiv \dfrac{1}{\sin^2 A \cos^2 A}$. This is an identity.

2. $\sin A = \cos A \tan A$.
3. $2 \sin^2 A - 1 = \sin^2 A - \cos^2 A$.
4. $\tan A = \sin A \sec A$.
5. $\sin A \cot A = \cos A$.
6. $\sqrt{1 - \cos^2 A} \cot A = \cos A$.
7. $(1 - \cos^2 A) \sec^2 A = \tan^2 A$.
8. $\dfrac{1 + \cot A}{1 + \tan A} = \dfrac{\cot A - 1}{1 - \tan A}$.
9. $\sin A \sec A \cot A = 1$.
10. $(\sin A + \cos A)^2 - 1 = 2 \sin A \cos A$.
11. $\sin A \cos A (\tan A + \cot A) - 1 = 0$.
12. $1 - 2 \csc^2 A = \cot^4 A - \csc^4 A$.
13. $\dfrac{1 + \sin A}{\cos A} = \dfrac{\cos A}{1 - \sin A}$.
14. $\tan^2 A \sin^2 A = \tan^2 A - \sin^2 A$.

519. Functions of an Angle in Terms of the Other Functions of the Same Angle.

Illustration.—To find $\tan \theta$ in terms of the other five functions of θ.

Solution.—$\tan \theta = \dfrac{\sin \theta}{\cos \theta}$. §518–2.

But $\cos \theta = \sqrt{1 - \sin^2\theta}$ and $\sin \theta = \sqrt{1 - \cos^2\theta}$.

Substituting, $\tan \theta = \dfrac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$. (1)

Also, $\tan \theta = \dfrac{\sqrt{1 - \cos^2\theta}}{\cos \theta}$. (2)

$\tan \theta = \dfrac{1}{\cot \theta}$. (3)

Now, $\tan^2\theta = \sec^2\theta - 1$ or $\tan \theta = \sqrt{\sec^2\theta - 1}$. (4)

Substituting $\dfrac{1}{\csc \theta}$ for $\sin \theta$ in (1) above we have,

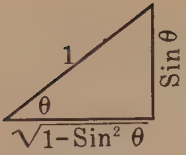
$$\tan \theta = \frac{\dfrac{1}{\csc \theta}}{\sqrt{1 - \dfrac{1}{\csc^2\theta}}} = \frac{\dfrac{1}{\csc \theta}}{\dfrac{\sqrt{\csc^2\theta - 1}}{\csc\theta}} = \frac{1}{\sqrt{\csc^2\theta - 1}}.$$

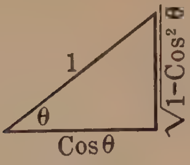
$$\therefore \tan \theta = \frac{1}{\sqrt{\csc^2\theta - 1}}. \qquad (5)$$

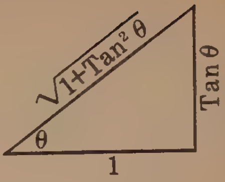
PROBLEMS

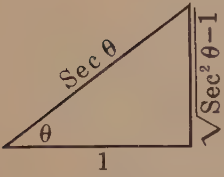
1. Find $\sin \theta$ in terms of the other five functions of θ.

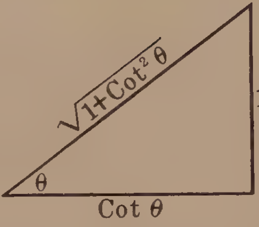
2. Find $\cos \theta$ and $\cot \theta$ in terms of the other five functions of θ. Any function of an angle in terms of the other functions of the same angle may be readily written from the following figures. These figures are drawn by the principle of §514.

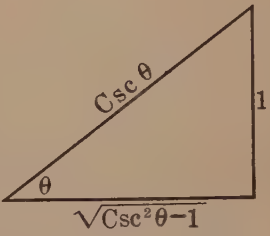




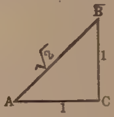






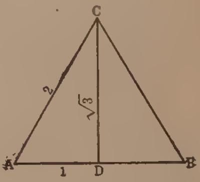


520. Functions of 30°, 45° and 60°.—Let ABC be an isosceles right triangle and each leg equal 1. Then the hypotenuse equals $\sqrt{2}$ and $\angle A = 45°$.



Write the six functions of 45°.

Let ABC be an equilateral triangle and each side equal 2, CD the altitude.



Then $\angle A = 60°$, $\angle ACD = 30°$, and $CD = \sqrt{3}$. Write the six functions of 30° and 60°.

These results, including those of §516, may be tabulated as follows:

Angle	sin	cos	tan	cot	sec	csc
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	1	0	∞	0	∞	1

PROBLEMS

Find the value of the angles in the following:

1. $\sin x = \frac{1}{2}$.
2. $\tan y = 1$.
3. $\sec \theta = \sqrt{2}$.
4. $\cos y = \frac{1}{2}\sqrt{2}$.
5. $\sin x = \frac{1}{2}\sqrt{3}$.
6. $\cot y = \dfrac{1}{\sqrt{3}}$.
7. $\csc \theta = \frac{2}{3}\sqrt{3}$.
8. $\tan \theta = \sqrt{3}$.
9. $\cos \theta = \frac{1}{2}$.
10. $\tan \theta = \infty$.
11. $\cos \theta = 1$.
12. $\sin \theta = 1$.
13. $\cot \theta = 0$.
14. $\cot \theta = \frac{1}{3}\sqrt{3}$.
15. $\cos x = 0$.

CHAPTER XIV

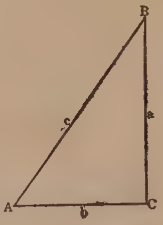
SOLUTION OF RIGHT TRIANGLES

521. **Solution of a Triangle.**—The solution of a triangle is the process of computing the values of all the unknown angles and sides of the given triangle. To solve a triangle three parts must be given, one of which must be a side. In this chapter only right triangles will be solved.

522. **Solution by Natural Functions.**—The student should first learn to solve right triangles by using natural functions. In §289 a short table of natural functions was given. It is now necessary to become familiar with the use of a more complete set of tables which will be explained in Chapter XVIII.

523. **The Four Cases of the Solution of Right Triangles.**—In solving a right triangle, the right angle is one of the given parts, therefore only two parts need be given. The two parts given come under four cases as follows:

1. The hypotenuse and one acute angle.
2. Either leg and one acute angle.
3. Both legs.
4. The hypotenuse and either leg.



When one acute angle is given or found, subtract it from 90° to get the other acute angle. The solution of each case will be illustrated using natural functions.

CASE I

Given $c = 25$ and $A = 50° 35'$. **Find** B, a, and b.
Solution.
1. $B = 90° - 50° 35' = 39° 25'$.
From the functions we have,

2. $\sin A = \frac{a}{c} \therefore a = c \sin A$.

From tables, $\sin A = .7725$
$c = 25$

Multiplying, $a = 19.3125$

3. $\cos A = \frac{b}{c} \therefore b = c \cos A$.

$\cos A = .6350$
$c = 25$

$\therefore b = 15.8750$

CASE II

Given $a = 42.5$ and $B = 25° 22'$. **Find** A, b, and c.
Solution.
1. $A = 90° - 25°22' = 64° 38'$.

2. $\tan B = \frac{b}{a} \therefore b = a \tan B.$

$$\tan B = .4741$$
$$a = 42.5$$
$$\therefore b = 20.14925$$

3. $\cos B = \frac{a}{c} \therefore c = \frac{a}{\cos B}.$

$$a = 42.5, \cos B = .9036$$
$$c = \frac{42.5}{.9036} = 47.03.$$

CASE III

Given $a = 30$ **and** $b = 62.5$. **Find** A, B, **and** c.
Solution.

1. $\tan A = \frac{a}{b}.$

$$\tan A = \frac{30}{62.5} = .4800$$
$$\therefore A = 25° 38' 30''$$

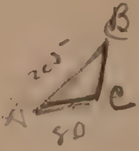
2. $B = 90° - A.$
$B = 90° - 25° 38' 30'' = 64° 21' 30''.$

3. $\sin A = \frac{a}{c} \therefore c = \frac{a}{\sin A}.$

$$a = 30, \sin A = .4328.$$
$$\therefore c = \frac{30}{.4328} = 69.3.$$

CASE IV

Given $c = 205$ **and** $b = 80$. **Find** A, B, **and** a.
Solution.



1. $\sin B = \frac{b}{c}.$

$$\sin B = \frac{80}{205} = .3902$$
$$B = 22° 58'$$

2. $A = 90° - B.$
$\therefore A = 90 - 22° 58' = 67° 2'.$

3. $\tan A = \frac{a}{b} \therefore a = b \tan A.$

$$b = 80$$
$$\tan A = 2.3597$$
$$\therefore a = 188.7760.$$

These results may be checked by the formula $a^2 + b^2 = c^2$.

PROBLEMS

Solve the following right triangles by Natural Functions.

	Given		Find
1.	$a = 12$,	$B = 38° 25'$.	b, c, and A.
2.	$a = 23.2$,	$A = 47° 27'$.	b, c, and B.
3.	$b = 2.3$,	$A = 53° 14'$.	a, c, and B.
4.	$c = 287$,	$B = 46° 15'$.	a, b, and A.
5.	$a = 8$,	$b = 6$.	c, A, and B.
6.	$a = 9.5$,	$c = 14.2$.	b, A, and B.
7.	$b = .384$,	$c = .542$.	a, A, and B.
8.	$b = 3.6$,	$B = 23° 47'$.	a, c, and A.
9.	$c = 425$,	$A = 61° 45'$.	a, b, and B.
10.	$c = .43$,	$b = .23$.	a, A, and B.
11.	$a = 2.53$,	$b = 2.60$.	c, A, and B.
12.	$c = .01$,	$a = .008$.	b, A, and B.
13.	$A = 18° 17' 30''$,	$c = 1$.	a, b, and B.
14.	$B = 15° 20'$,	$b = .1$.	a, c, and A.
15.	$a = 2b$,	$c = 25$.	b, A, and B.

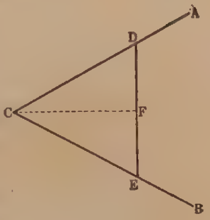
16. Measure the angle ACB using a tape.

Hint.—*Measure along* CA 200 *feet to* D *and along* CB 200 *feet to* E. *Measure* DE. *Suppose* DE *equals* 175 *feet. Draw* ⊥ *from* C *to* DE *as* CF. *Then* $\sin \frac{1}{2}C = \frac{\frac{1}{2}DE}{200} = \frac{DE}{400}$. *Find* $\angle C$.

17. Suppose CD measures 100 feet, and DE 140 feet, find $\angle C$.

18. A train is moving south 50° west at a rate of 30 miles an hour. At what rate is it moving due south; at what rate due west? See §550.

19. At a distance of 100 feet from the base of a tower, the angle of elevation of the top is observed to be 40° 20′. Find the height.



PROB. 16.

524. Solution by Logarithms.—Before proceeding further with this chapter the student must learn how to use logarithms. The treatise on this subject will be found in Chapter XVIII. The use of logarithms is almost indispensable in engineering work.

525. Solution of Right Triangles by Logarithms.

Illustration.

CASE I

Given $c = 25.42$ **and** $B = 43° 20' 10''$. **Find** A, a, **and** b.

Solution.

1. $A = 90° - 43° 20' 10'' = 46° 39' 50''$.

2. $\cos B = \frac{a}{c}$.

$$\therefore a = c \cos B.$$

$$\begin{aligned} \log c &= 1.40518 \\ \log\cos B &= 9.86174 \\ \hline \log a &= 1.26692 \\ a &= 18.489. \end{aligned}$$

3. $\sin B = \frac{b}{c}$.

$$\therefore b = c \sin B.$$

$$\begin{aligned} \log c &= 1.40518 \\ \log\sin B &= 9.83650 \\ \hline \log b &= 1.24168 \\ b &= 17.445. \end{aligned}$$

Check

$$a^2 + b^2 = c^2 \therefore a^2 = c^2 - b^2 = (c + b)(c - b).$$

$$\begin{aligned} c &= 25.42 \\ b &= 17.445 \\ c + b &= 42.865 \\ c - b &= 7.975. \end{aligned}$$

$$\begin{aligned} \log (c + b) &= 1.63210 \\ \log (c - b) &= 0.90173 \\ \hline \log a^2 &= 2.53383 \\ \log a &= 1.26692 \end{aligned}$$

which checks with **log a** above.

Note.—*The results obtained in work by logarithms will be more accurate if the treatment of* .5 *is uniform. Thus, if we have* 3.5 *make it* 4 *and if we have* 6.5 *make it* 6. *In other words always take the nearest even number.*

CASE II

Given $b = 384.63$ **and** $A = 32° 28' 30''$. **Find** B, a, **and** c.

Solution.

1. $B = 90° - 32° 28' 30'' = 57° 31' 30''$.

2. $\tan A = \dfrac{a}{b}.$

$$\therefore a = b \tan A.$$

$$\begin{aligned} \log b &= 2.58504 \\ \log \tan A &= 9.80377 \\ \hline \log a &= 2.38881 \\ a &= 244.80. \end{aligned}$$

3. $\cos A = \dfrac{b}{c}.$

$$\therefore c = \frac{b}{\cos A}.$$

$$\begin{aligned} \log b &= 2.58504 \\ \log \cos A &= 9.92615 \\ \hline \log c &= 2.65889 \\ c &= 455.92. \end{aligned}$$

Check

$$\begin{aligned} c &= 455.92 \\ b &= 384.63 \\ \hline c + b &= 840.55 \\ c - b &= 71.29. \end{aligned}$$

$$\begin{aligned} \log (c + b) &= 2.92456 \\ \log (c - b) &= 1.85303 \\ \hline \log a^2 &= 4.77759 \\ \log a &= 2.38880. \end{aligned}$$

This is a difference of **1** from **log a** above which is a very good check.

CASE III

Given $a = .38654$ and $b = .60235$. **Find** A, B, and c.

Solution.

1. $\tan A = \dfrac{a}{b}.$

$$\begin{aligned} \log a &= 9.58719 \\ \log b &= 9.77985 \\ \hline \log \tan A &= 9.80734 \\ \therefore A &= 32°41'19''. \end{aligned}$$

2. $B = 90° - A.$

$$\therefore B = 90° - 32°\ 41'\ 19'' = 57°\ 18'\ 41''.$$

3. $\sin A = \dfrac{a}{c}.$ $\quad \therefore c = \dfrac{a}{\sin A}.$

$$\begin{aligned} \log a &= 9.58719 \\ \operatorname{logsin} A &= 9.73245 \\ \hline \log c &= 9.85474 \\ \therefore c &= .71572. \end{aligned}$$

Check

$$\begin{aligned} c &= .71572 \\ b &= .60235 \\ \hline c + b &= 1.31807 \\ c - b &= .11337. \end{aligned}$$

$$\begin{aligned} \log (c + b) &= 0.11994 \\ \log (c - b) &= 9.05450 \\ \hline \log a^2 &= 9.17444 \\ \log a &= 9.58722. \end{aligned}$$

This is a difference of 3 from **log a** above which is acceptable as a check.

CASE IV

Given $c = 38462$ and $b = 20359$. **Find** A, B, and a.

Solution.

1. $\sin B = \dfrac{b}{c}.$

$$\begin{aligned} \log b &= 4.30876 \\ \log c &= 4.58503 \\ \hline \text{logsin } B &= 9.72373 \\ \therefore B &= 31^\circ 57' 37''. \end{aligned}$$

2. $A = 90^\circ - B.$

$$\therefore A = 90^\circ - 31^\circ\ 57'\ 37'' = 58^\circ\ 2'\ 23''.$$

3. $\cos B = \frac{a}{c}. \quad \therefore a = c \cos B.$

$$\begin{aligned} \log c &= 4.58503 \\ \log \cos B &= 9.92861 \\ \hline \log a &= 4.51364 \\ \therefore a &= \quad 32632. \end{aligned}$$

Check

$$\begin{aligned} c &= 38462 \\ b &= 20359 \\ \hline c + b &= 58821 \\ c - b &= 18103. \end{aligned}$$

$$\begin{aligned} \log (c + b) &= 4.76954 \\ \log (c - b) &= 4.25775 \\ \hline \log a^2 &= 9.02729 \\ \log a &= 4.51364. \end{aligned}$$

This checks exactly with log a above.

PROBLEMS

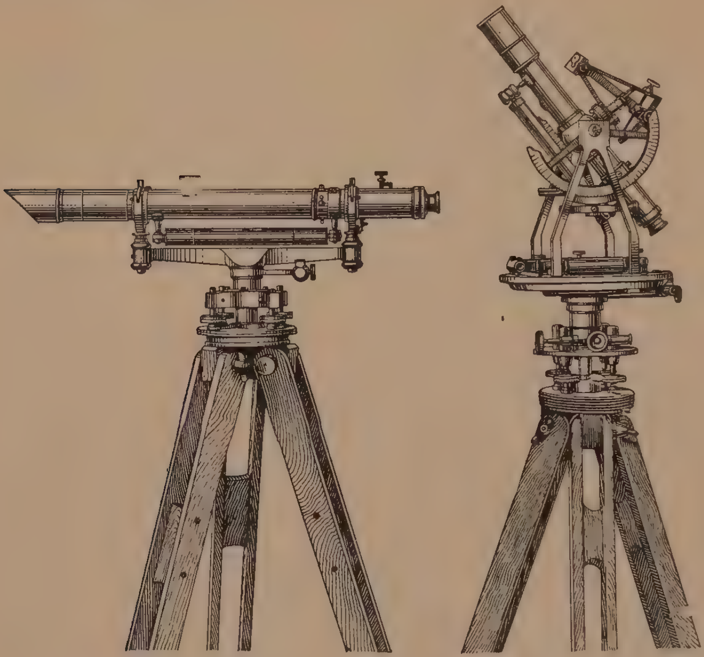
Solve the following right triangles by the use of logarithms.

	Given		Find
1.	$c = 16,$	$A = 25^\circ 50'.$	a, b, and B.
2.	$c = 38.42,$	$B = 35^\circ 22'.$	a, b, and A.
3.	$a = 62.54,$	$B = 48^\circ 35'.$	b, c, and A.
4.	$b = 725,$	$A = 22^\circ 47'.$	a, c, and B.
5.	$a = 3.1416,$	$b = 2.8750.$	c, A, and B.
6.	$a = 99.462,$	$c = 156.820.$	b, A, and B.
7.	$b = .3252,$	$c = .5384.$	a, A, and B.
8.	$A = 38^\circ 25' 40'',$	$b = .038462.$	a, c, and B.
9.	$B = 51^\circ 2' 13'',$	$a = 2.3843.$	b, c, and A.
10.	$a = 68432,$	$b = 53281.$	c, A, and B.
11.	$b = .84325,$	$c = 1.23840.$	a, A, and B.
12.	$a = 4.3827,$	$c = 8.3789.$	b, A, and B.
13.	$A = 31^\circ 45' 15','$	$a = 723.15.$	b, c, and B.
14.	$a = .003864,$	$c = .006031.$	b, A, and B.
15.	$a = 6.72385,$	$b = 5.8642.$	c, A, and B.

Solve the following isosceles triangles in which A and B are the equal angles; a, b, and c the sides opposite A, B, and C respectively, and h the altitude.

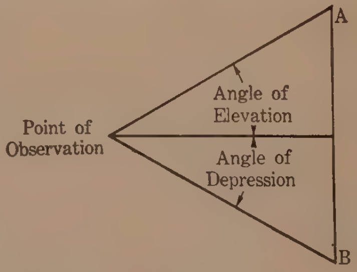
	Given		Find
16.	$A = 62^\circ 45',$	$b = 25.472.$	c, h, and C.
17.	$B = 37^\circ 42' 35'',$	$c = 425.38.$	b, h, and C.
18.	$h = 45.268,$	$c = 80.482.$	a, A, and C.
19.	$a = .08245,$	$c = .09372.$	h, A, and C.
20.	$C = 72^\circ 28' 36'',$	$h = 3.8645.$	a, c, and A.
21.	$C = 50^\circ 38' 44'',$	$a = 3864.5.$	h, c, and A.

22. A regular hexagon is inscribed in a circle whose diameter is 10. Find the radius of the inscribed circle.



Surveyor's level and transit. (*From Barker's Plane Trigonometry.*)

23. Find the height of a tower as observed from the summit of a hill if the angle of elevation to the top is 30°, and the angle of depression to the base is 35°. The horizontal distance from the point of observation to the tower is 210 feet.



24. Find the angle of elevation of the sun when a tower 550.25 feet high casts a shadow 725.32 feet in length.

25. Find the length of a diagonal of a regular pentagon whose side is 3.8642.

26. Find the angle of elevation of a mountain slope which rises 472.8 feet in a horizontal distance of one fourth of a mile.

27. The radius of a circle is 10.384 feet. Find the side and the apothem of a regular inscribed decagon.

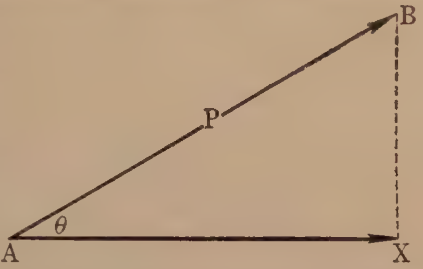
28. The angle of elevation of the sun is 55° 25′ 30″. The length of the shadow of a tree on a level plane is 48.382 feet. Find the height of the tree.

29. The length of a kite string is 750 feet, and the angle of elevation of the kite is 38°45′. Supposing the kite string to be straight, find the height of the kite.

30. At a point a certain distance from the Washington Monument, which is 555 feet high, the angle of elevation to the top was found to be 47° 25′. The observer moved back in a straight line from the monument and then found the angle of elevation to be 24°38′. How far did the observer move from the first point of observation?

31. If a hillside has a slope of 9°40′, how far back will a dam 15.32 feet high force the water?

32. *AX* in the figure represents the component of the force *P*. If θ is the angle between the direction of *P* and that of *AX*, the magnitude of the component is *P* cos θ. What is the component *AX* when the force is 200 pounds and the angle θ is 28° 20′.



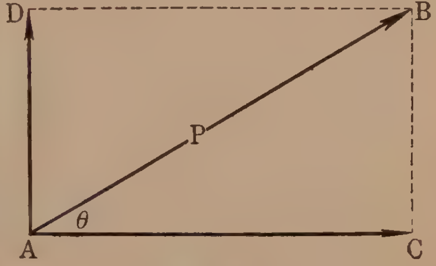
33. In the same figure, find the force when the component is 125 pounds and θ is 39° 50′.

34. Find the angle θ when the force is 500 lbs. and the component is 285.5 lbs.

35. Suppose the force (*P*) is 18.2 pounds and acts northward, compute the component of *P* which acts in a northwest direction. *Ans.* 12.87

36. Horizontal and vertical forces are illustrated by the figure. If *AB* represents the magnitude of *P*, then it is found that the horizontal force is represented by *AC* and the vertical force by *AD*, and these represent the horizontal and vertical components of *P*. What horizontal and vertical forces must be applied to the body at *A* in order to prevent motion, if *P* is 100 pounds and θ is 44° 45 ?

Ans. Horizontal = 71.02 and vertical = 70.40.



PROB. 36.

37. Compute the horizontal and vertical components of a force of 10,000 pounds when it makes an angle of 75° 30′ with a horizontal line.

38. In a horizontal plane a force of 67.5 pounds acts eastward and is held in equilibrium by two equal forces acting northwestward and southwestward. Find each of these forces.

39. In the same figure the horizontal component is 150 pounds and angle θ is 35°22′30″, find the force (*P*) and the vertical component.

40. Suppose the horizontal component is 385.2 pounds and the vertical component is 405.5 pounds, find the force (*P*) and also the angle θ.

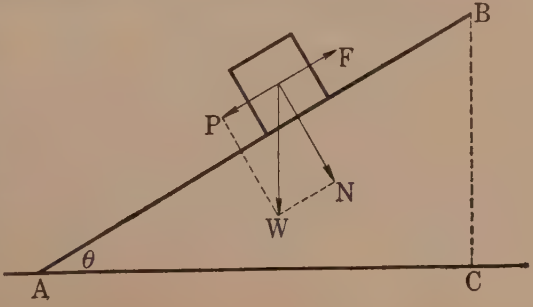
41. Find the horizontal component and angle θ if the vertical component is 3842.5 pounds and the force is 5648.3 pounds.

42. If the force (*P*) is 1000 kilograms and θ is 240°, what are the horizontal and vertical components?

43. A body weighing 20 pounds rests on a table and is acted upon by a force of 80 pounds making an angle of 30°15′ with the horizontal. What is the total vertical pressure on the table? *Ans.* 60.302 lbs.

44. If the horizontal and vertical components of a force are −64.8 and +108.4 pounds, what is the magnitude of the force? What angle does it make with the horizontal?

45. The coefficient of friction is indicated in the figure where the plane *AB* is slowly rising around the hinge A until the body *P* begins to slide. If θ is the angle *BAC*, the normal (perpendicular) pressure *N* is *W* cos θ and the resisting friction *F* along the plane is equal to *W* sin θ. Accordingly the coefficient of friction is **n**.



Hence, $n = \frac{F}{N} = \frac{W \sin \theta}{W \cos \theta} = \tan \theta$, i. e., the coefficient of friction is equal to the tangent of the angle BAC. For example, if a brick begins to slide on a board when the board is inclined 16°15′ to the horizontal, the coefficient of friction is $n = \tan 16° 15' = 0.29147$.

46. A body weighing 52 pounds rests upon a table, and a force of 26 pounds acts horizontally upon it. If sliding just begins, compute the coefficient of friction.

47. A body begins to slide down a plane when it is inclined 35°15′ to the horizontal. What is the coefficient of friction for the surfaces of contact?

48. If the coefficient of friction is 0.2056 at what angle must the plane be inclined so that sliding begins?

49. If in the figure of problem 45, F equals 28.52 and N equals 36.28, find the angle the plane is inclined to the horizontal.

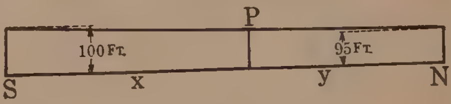
50. If N is 39.465 and θ is 35° 42′ 35″, find F and W.

51. If θ is 25° 32′ 30″ and W is 384.62, find n, F and N.

52. The height of the Woolworth Building in New York City is 750 feet. A man walks away from this building so that the angle of elevation is 50°25′. How far did he walk away?

53. From the top of a mountain two consecutive mile-stones are observed on a straight horizontal road leading from the base of the mountain. The angles of depression were observed to be 15° 25′ and 8° 32′ respectively. Find the height of the mountain.

54. A street is 900′ long (measured horizontally). At the south end the elevation is 100′ and the grade is down .6%. At the north end the elevation is 95′ and the grade is up .5%. Find the elevation and location of the point P, where the grade changes.



PROB. 54.

Hint. $100 - .006x = 95 + .005y$.

Note.—*A grade of 1% means a rise or fall of 1 foot per 100 feet in a horizontal distance. Hence the tangent of the angle which a 1% grade line makes with the horizontal is .01. The tangent of the angle of a 2.5% grade line with the horizontal is .025.*

55. The elevation of the east end of a street 500′ long is 48.25′ and of the west end 66.47′. The grade from the east end is up 4.5%, and from the west end the grade is down 2.5%. Find the elevation and location of the point where the grade changes. *Ans.* Elevation 61.12′. Distance 286′ from east end.

56. The grade elevation at the north end of a street 5408 feet long is 12.33 feet and the south end it is 9.21 feet. The grade from the south end to the summit is 2.25%; from the north end to the summit it is 1.15%. Find the location and elevation of the summit. *Ans.* Elev. 52.43 feet. 1920.94 feet from the south end.

57. It is required to find the distance from a point B to an inaccessible point A. The angle ABC was laid off 90° and the angle was BCA measured 45° 48′. BC was measured and found to be 120.85 feet. Find AB. (Draw figure.)

Ans. 124.27 feet.

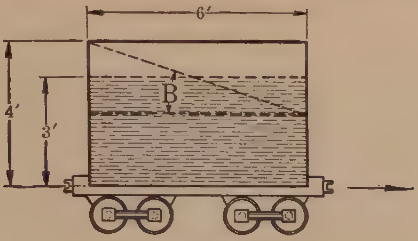
58. It is given in geography that the length of a degree of longitude on the equator is about 69.16 miles. (*a*) Derive a general formula for finding the length of a degree of longitude at any latitude. (*b*) Find the length of a degree of longitude at 30° N latitude. (*c*) At 50° S. latitude.

Ans. (*a*) 69.16 cos of the angle of latitude.

59. Find the area of a segment of a circle, if the height of the segment is 2 feet and its chord is 8 feet. **Hint.**—*Apply* §322 *to find the diameter of the circle.*

60. What is the radius of a circle in which a 100 foot chord subtends an angle of 12° 27′ at a point on the circumference?

61. A locomotive tender contains a water tank 6 feet square and 4 feet deep. If the water stands within 1 foot of its top, what maximum uniform acceleration of the train may be permitted without losing water from the tank?



PROB. 61.

The tangent of the angle the water surface makes with the horizontal equals the acceleration divided by gravity. i.e. $\tan B = \frac{\text{acc}}{\text{gravity}}$. $\text{Tan } B = \frac{2}{6}$, gravity $= 32.2$.

.72

Ans. 10.7′ per sec.

62. A flag pole is made of two sections. The lower section is 20 feet and the upper section is 30 feet long. How far from the foot of the pole in the same plane as the base will the two sections subtend equal angles?

63. A tower 100 feet high stands on a building 75 feet high. At what distance from the building will the tower and the building subtend equal angles at a point 5 feet above the plane of the base of the building?

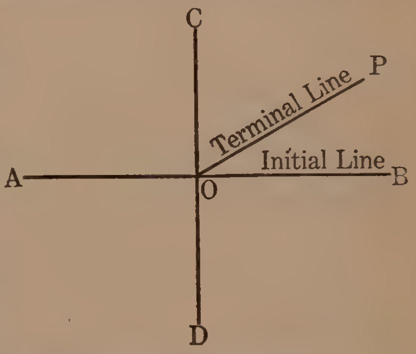
CHAPTER XV

FUNCTIONS OF ANGLES IN GENERAL

526. **Angles Greater than 90°.**—Before oblique triangles can be solved, we must extend the definitions of the functions of acute angles to the functions of angles of any magnitude. In fact we should be able to find the functions of any angle.

527. **The Four Quadrants.**—Let *CD* be perpendicular to *AB* and intersect *AB* at *O*. Positive angles are generated by rotating the line *OP* counter-clockwise or in the direction opposite to the movement of the hands of a clock. Negative angles are generated by rotating *OP* clockwise.

In the first quadrant the angles are between 0° and 90°; in the second quadrant they are between 90° and 180°; in the third quadrant they are between 180° and 270°; and in the fourth quadrant they are between 270° and 360°. It will be understood that the angle always starts at the line *OB*. Hence, when *OP* coincides with *OB* the angle will be 0°. The line *OB* is the initial line and *OP*, (*P* may be in any quadrant) the terminal line. As has been shown in graphs (§340) the line *AB* is the axis of abscissas and *CD* the axis of ordinates. The origin is the point *O* where the axes *AB* and *CD* intersect. The abscissa of a point is the distance of the point to the right or left of the *CD* axis. The ordinate of a point is the distance of the point above or below the *AB* axis.



528. **Functions of Any Angle.**—In §289 functions of acute angles were defined. These definitions must now be extended to include the functions of angles of any magnitude.

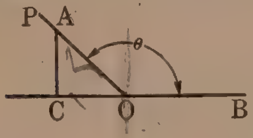


Fig. 1.

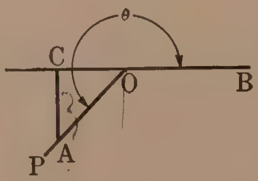


Fig. 2.

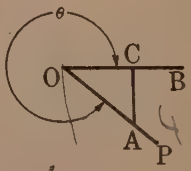


Fig. 3.

In figures (1), (2), and (3), let *OB* be the initial line and *OP* the terminal line of an angle θ which may be in the second, third, or fourth quadrant. From A, any point in OP draw AC perpendicular to OB or BO extended. In each of the three figures, (a) $\sin\theta = \frac{AC}{OA}$, (b) $\cos\theta = \frac{OC}{OA}$, (c) $\tan\theta = \frac{AC}{OC}$.

Hence, if from any point in the terminal line of an angle, a perpendicular is drawn to the initial line or the initial line extended we have, (a) The sine of an angle of any magnitude is the ratio of the length of the perpendicular to the distance from the vertex of the angle to the top of the perpendicular.

(b) The cosine of an angle of any magnitude is the ratio of the distance from the vertex to the foot of the perpendicular, to the distance from the vertex of the angle to the top of the perpendicular.

(c) The tangent of an angle of any magnitude is the ratio of the length of the perpendicular, to the distance from the vertex of the angle to the foot of the perpendicular.

The cotangent, the secant, and the cosecant of any angle may be derived from the reciprocal relations.

529. **The Signs of the Functions in the Four Quadrants.**—Let CD be drawn perpendicular to AB and intersect AB at O. Let OB be the common initial side for angles in the four quadrants, and O the common vertex. Let y_1, y_2, y_3, and y_4 be perpendiculars (**ordinates**) to AB from any point P on the terminal line. x_1, x_2, x_3, and x_4 are the **abscissas** of P. OP (c) is considered positive in any position. θ_1, θ_2, θ_3, and θ_4 are angles in the first, second, third, and fourth quadrants respectively.

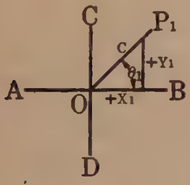


FIG. 1.

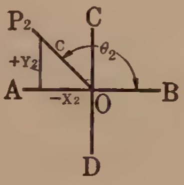


FIG. 2.

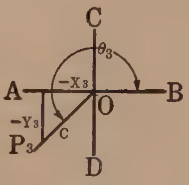


FIG. 3.

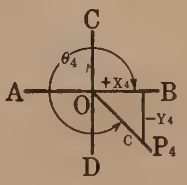


FIG. 4.

From Fig. 1.	From Fig. 2.	From Fig. 3.	From Fig. 4.
$\sin\theta_1 = \frac{y_1}{c}$.	$\sin\theta_2 = \frac{y_2}{c}$.	$\sin\theta_3 = \frac{-y_3}{c}$.	$\sin\theta_4 = \frac{-y_4}{c}$.
$\cos\theta_1 = \frac{x_1}{c}$.	$\cos\theta_2 = \frac{-x_2}{c}$.	$\cos\theta_3 = \frac{-x_3}{c}$.	$\cos\theta_4 = \frac{x_4}{c}$.
$\tan\theta_1 = \frac{y_1}{x_1}$.	$\tan\theta_2 = \frac{y_2}{-x_2}$.	$\tan\theta_3 = \frac{-y_3}{-x_3}$.	$\tan\theta_4 = \frac{-y_4}{x_4}$.

The cotangent, the secant, and the cosecant of an angle in any quadrant can be derived from the above figures, or they may be derived from the reciprocal relations given in §518.

From the above ratios we have:

1. **Sin** θ *and* **csc** θ *are positive in the first and second quadrants, and negative in the third and fourth quadrants.*

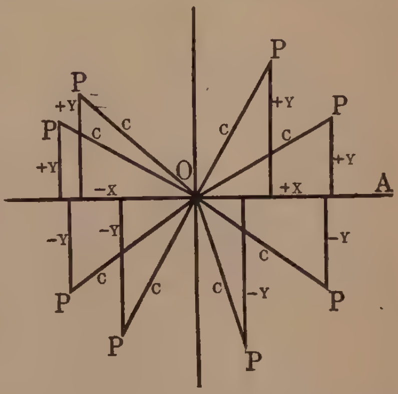
2. **Cos** θ *and* **sec** θ *are positive in the first and fourth quadrants, and negative in the second and third quadrants.*

3. **Tan** θ *and* **cot** θ *are positive in the first and third quadrants, and negative in the second and fourth quadrants.*

The signs of the functions in the four quadrants are easily remembered from the accompanying figure.

sin and csc + *other functions −*	*all functions +*
tan and cot + *other functions −*	*cos and sec +* *other functions −*

530. **Changes in the Values of the Functions as an Angle Increases from 0° to 360°.**—In the figure let OA be the initial line, OP the terminal line, and O the vertex of an angle θ as it increases from 0° to 360°. Let $\mathbf{y}$ be the ordinate, $\mathbf{x}$ the abscissa, and $\mathbf{c} = OP$.



By a careful study of the figure the following changes in the values of the various functions in the four quadrants will be readily seen.

1. **Sin** θ.—In the first quadrant $\mathbf{y}$ increases from 0 to $\mathbf{c}$, therefore the ratio $\frac{y}{c}$ or sin θ increases from 0 to 1.

In the second quadrant $\mathbf{y}$ decreases from $\mathbf{c}$ to 0, therefore the ratio $\frac{\mathbf{y}}{c}$ or sin θ decreases from 1 to 0.

In the third quadrant $\mathbf{y}$ increases in absolute value from 0 to $\mathbf{c}$, but since $\mathbf{y}$ is negative the ratio $\frac{-\mathbf{y}}{c}$ or sin θ decreases from 0 to -1.

In the fourth quadrant $\mathbf{y}$ decreases in absolute value from $\mathbf{c}$ to 0 but again $\mathbf{y}$ is negative and the ratio $\frac{-\mathbf{y}}{c}$ or sin θ increases from -1 to 0.

2. **Cos** θ.—Following the same reasoning as in (1) we have the following: In the first quadrant cos θ decreases from 1 to 0; in the second, it decreases from 0 to -1; in the third, it increases from -1 to 0; and in the fourth, it increases from 0 to 1.

3. **Tan** θ.—In the first quadrant tan θ increases from 0 to $+\infty$ (an infinite value); in the second, it increases from $-\infty$ to 0; in the third it increases from 0 to $+\infty$; and in the fourth it increases from $-\infty$ to 0.

4. **Cot** θ, **sec** θ, **csc** θ.—The changes in these functions can be traced from the figure or they may be derived by use of the reciprocal relations given in §518.

531. **Limiting Values of the Functions.**—It will be seen from the foregoing discussion that the sine and cosine of an angle can have any value between 1 and -1. They cannot have any value greater than 1 or less than -1.

The tangent and cotangent can have any value positive or negative whatsoever.

The secant and cosecant can have any value greater than 1 and less than −1, but cannot have any value between 1 and −1.

532. **Short Table of Natural Functions.**—The following table shows the values of the functions of the angles most often used in engineering work.

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{3}$	-1
tan	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{3}\sqrt{3}$	0

PROBLEMS

1. Determine the quadrants in which the following angles are found:

(*a*) 125°. (*b*) 175°. (*c*) 225°. (*d*) 135°. (*e*) 425°.
(*f*) 1000°. (*g*) −25°. (*h*) −335°. (*i*) −285°. (*j*) −180°.

2. Construct the functions of an angle in the second quadrant. What are their signs?

3. Do the same for the third and fourth quadrants.

4. In which quadrant does an angle lie: (*a*) If the tangent is negative and the sine positive? (*b*) If the cosine is negative and the cotangent positive? (*c*) If the secant is positive and the tangent negative? (*d*) If the tangent is negative and the cosine negative? (*e*) If the cosecant is positive and the cotangent negative? (*f*) If all the functions are positive?

5. What are the signs of the functions of the following angles: 138°; −120°; −300°; −25°; −135°; 135°; 120°; 150°?

6. How many angles less than 360° have the value of the tangent equal to $\frac{3}{2}$? In what quadrants do they lie?

7. How many values less than 180° can x have if: (*a*) $\sin x = \frac{2}{3}$? (*b*) $\cot x = -\frac{4}{3}$? (*c*) $\sec x = -\frac{5}{3}$? (*d*) $\cos x = \frac{3}{4}$?

8. In finding $\sin x$ by the equation, $\sin x = \pm\sqrt{1 - \cos^2 x}$, when must we choose the positive sign; when the negative sign?

9. Given $\sec x = -2$, find the other functions of **x** when **x** is in the third quadrant. (Draw a figure.)

10. Given $\cot y = -3$, find the other functions of **y** when **y** is in the fourth quadrant.

11. Given $\sin y = \frac{3}{4}$, find all the other functions of **y**.

12. Given $\tan x = 2$, find all the other functions of **x**.

13. Find the value of $3 \sin 90° - 5 \sec 180°$.

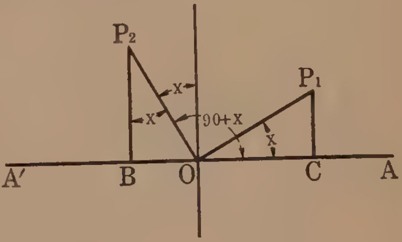
14. Find the value of $b \tan 0° - b \sec 360° + b \cos 180°$.

15. Find the value of $2 \cot 90° + 3 \cos 0° + 2 \sin 270° - 6 \tan 360°$.

533. **Functions of 90° + x, 180° ± x, 270° ± x, 360 − x, and − x, in Terms of ∠ x, where x is Acute.**—In the solution of oblique triangles

it is often necessary to use the functions of an angle between 90° and 180°. It is convenient to find the functions of such angles in terms of an angle less than 90°.

The functions of $90° + x$, $180° - x$, and of $-x$, will be derived. The others may be derived in a similar way if desired.



I. Functions of 90 + x in Terms of x.—Let $\angle\ AOP_2$ be the angle $90° + x$.

Construct $\angle\ x$ in the first quadrant.

Make $OP_1 = OP_2$ and draw P_1C and $P_2B \perp A'A$.

Now rt. $\triangle\ BOP_2 =$ rt. $\triangle\ OCP_1$. §137.

$\therefore\ P_2B = OC$ and $BO = P_1C$. Why?

$\sin (90° + x) = \frac{P_2B}{OP_2}$, $\cos x = \frac{OC}{OP_1}$. But $\frac{P_2B}{OP_2} = \frac{OC}{OP_1}$.

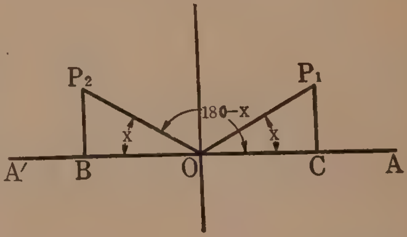
Hence, **sin (90° + x) = cos x.**

$\cos (90° + x) = \frac{OB}{OP_2}$, $\sin x = \frac{P_1C}{OP_1}$. But $\frac{OB}{OP_2} = -\frac{P_1C}{OP_1}$, for $OB = -P_1C$.

Hence, **cos (90° + x) = −sin x.**

$\tan (90° + x) = \frac{P_2B}{OB}$, $\cot x = \frac{OC}{P_1C}$. But $\frac{P_2B}{OB} = -\frac{OC}{P_1C}$.

Hence, **tan (90 + x) = −cot x.**



II. Functions of 180° − x in Terms of x.—Let $\angle\ AOP_2$ be the angle $180° - x$.

Construct $\angle\ x$ in the first quadrant.

Make $OP_1 = OP_2$ and draw P_1C and $P_2B \perp A'A$.

Now rt. $\triangle\ BOP_2 =$ rt. $\triangle\ OCP_1$. Why?

$\therefore\ OB = OC$ and $P_2B = P_1C$. Why?

$\sin (180° - x) = \frac{P_2B}{OP_2}$, $\sin x = \frac{P_1C}{OP_1}$. But $\frac{P_2B}{OP_2} = \frac{P_1C}{OP_1}$.

Hence, **sin (180° − x) = sin x.**

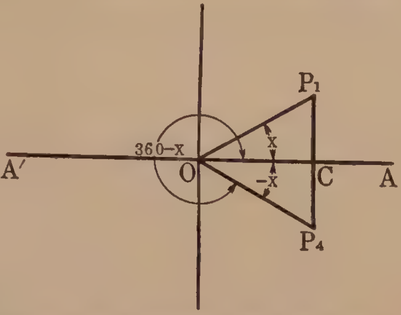
$\cos (180° - x) = \frac{OB}{OP_2}$, $\cos x = \frac{OC}{OP_1}$. But $\frac{OB}{OP_2} = -\frac{OC}{OP_1}$.

Hence, **cos (180° − x) = −cos x.**

$\tan (180° - x) = \frac{P_2B}{OB}$, $\tan x = \frac{P_1C}{OC}$. But $\frac{P_2B}{OB} = -\frac{P_1C}{OC}$.

Hence, **tan (180° − x) = −tan x.**

III. Functions of $(-x)$ or $360° - x$ in Terms of x.—In the figure, rt. $\triangle OCP_1$ = rt. $\triangle OCP_4$. Why?



$$\sin(-x) = \frac{CP_4}{OP_4}, \quad \sin x = \frac{P_1C}{OP_1}.$$

But $\dfrac{CP_4}{OP_4} = -\dfrac{P_1C}{OP_1}$.

Hence, $\sin(-x) = -\sin x$.

In the same way.

$\cos(-x) = \cos x$, and $\tan(-x) = -\tan x$.

The cotangent, the secant, and the cosecant of the angles in I, II, and III may be derived from the figure or from the reciprocal relations.

The above may be briefly summarized as follows:

1. *The functions of* $90° + x$, $270 \pm x$ *are equal to the co-functions of x. The sign is determined by the quadrant in which the angle lies.*

2. *The functions of* $180° \pm x$ *and* $360° - x$ *or* $(-x)$ *are equal to the same functions of x. The sign is determined by the quadrant in which the angle lies.*

The following formulas are given for reference.

$\sin(90° + x) = \cos x.$
$\cos(90° + x) = -\sin x.$
$\tan(90° + x) = -\cot x.$
$\cot(90° + x) = -\tan x.$

$\sin(180° + x) = -\sin x.$
$\cos(180° + x) = -\cos x.$
$\tan(180° + x) = \tan x.$
$\cot(180° + x) = \cot x.$

$\sin(270° + x) = -\cos x.$
$\cos(270° + x) = \sin x.$
$\tan(270° + x) = -\cot x.$
$\cot(270° + x) = -\tan x.$

$\sin(180° - x) = \sin x.$
$\cos(180° - x) = -\cos x.$
$\tan(180° - x) = -\tan x.$
$\cot(180° - x) = -\cot x.$

$\sin(270° - x) = -\cos x.$
$\cos(270° - x) = -\sin x.$
$\tan(270° - x) = \cot x.$
$\cot(270° - x) = \tan x.$

$\sin(-x) = -\sin x.$
$\cos(-x) = \cos x.$
$\tan(-x) = -\tan x.$
$\cot(-x) = -\cot x.$

PROBLEMS

1. Express cos 250° in terms of the functions of an acute angle.

Solution.—(1) $\cos 250° = \cos(180° + 70°) = -\cos 70°$. (2) $\cos 250° = \cos(270° - 20°) = -\sin 20°$.

Express the following functions in terms of the functions of an acute angle.

2. sin 135°.
3. cos 300°.
4. tan 100°.
5. cot 220°.
6. cos 145°.
7. sin 340°.
8. cot 140° 25′.
9. tan 195° 38′.
10. sin 299° 40′.

Find the numerical value of the following:

11. sin 135°.
12. cos 225°.
13. tan 150°.
14. cot 330°.
15. cos 300°.
16. sin (−30°).
17. tan (−225°).
18. tan 120°.
19. cot 150°.
20. sin 120°.
21. sin 150°.
22. cos 135°.

23. Reduce the functions of the following angles to the functions of positive angles less than 45°: 65°; 175°; 138°; 95° 22′; 50° 22′; and 130° 52′.

24. Find the numerical values of all the functions of the following angles: 225°; 150°; 210°; 135°; −30°; and −120°.

25. Given $\cos x = -\frac{1}{2}$ and $\sin x$ positive. Find the other functions of x and the value of x.

26. Given $\tan x = -\sqrt{3}$ and x in the second quadrant. Find the other functions of x and the value of x.

27. Find all the values of x less than 360°: (*a*) If $\tan x = \sqrt{3}$. (*b*) If $\sin x = \frac{1}{2}\sqrt{2}$. (*c*) If $\cos x = -\frac{1}{2}$.

28. Find the value of $3 \sin (90° + x) - 2 \cos (180° - x)$.

29. Find the value of $\cos (180° + x) \sin (270° - x)$.

30. Evaluate $(a - b) \tan (90° + x) - (a + b) \cot (180° - x)$. *Ans.* $2b \cot x$.

31. Evaluate $5 \tan (270° + x) + 6 \tan (270° - x)$. *Ans.* $\cot x$.

32. Evaluate $\sin (270° + x) \cos (180° - x) - \cos (180° - x) \sin (270° - x)$. *Ans.* 0

Find the value of θ in each of the following equations.

33. $\cos^2\theta - \sin\theta = \sin^2\theta$.

Solution.—$\cos^2\theta = 1 - \sin^2\theta$.

Substituting, $1 - \sin^2\theta - \sin\theta = \sin^2\theta$.

Collecting, $-2 \sin^2\theta - \sin\theta + 1 = 0$.

$$\text{or } 2 \sin^2\theta + \sin\theta - 1 = 0.$$

Factoring, $(2 \sin\theta - 1)(\sin\theta + 1) = 0$.

Solving, $\therefore \sin\theta = \frac{1}{2}$ or -1.

If $\sin\theta = \frac{1}{2}$, then $\theta = 30°$ or 150°. These values are found in the table of §532. If $\sin\theta = -1$, then $\theta = 270°$.

34. $\cos\theta = \sin\theta$.

35. $\tan^2\theta = 3$.

36. $\sin^2\theta = \frac{3}{4}$.

37. $2 \sin^2\theta + \cos^2\theta = 2$.

38. $\sec^2\theta - 2 = 0$.

39. $\tan\theta + \cot\theta = 2$.

40. $2 \sec\theta - \cos\theta = 1$.

41. $\sin^2\theta - \frac{1}{4} = \cos\theta$.

42. $\cos^2\theta = 1.3 \sin^2\theta$.

43. $\sqrt{2} \tan\theta = \sec\theta$.

44. $2 \sin\theta = \tan\theta$.

45. $\sin^2\theta - 3 \cos^2\theta + \tan^2\theta = 0$.

534. Functions of the Sum of Two Angles.—To shorten the algebraic work, let the terminal lines of the angles equal 1. It must be remembered however, that the functions of angles should always be considered as ratios rather than as lines.

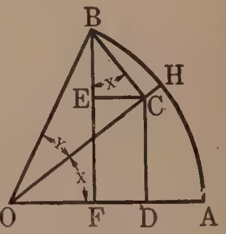


Fig. 1.

Let $\angle AOH = x$, $\angle HOB = y$, $BF \perp OA$, $BC \perp OH$, $CE \perp BF$, and $CD \perp OA$.

Then $\angle AOB = \angle x + \angle y$. Also $\angle x = \angle CBE$. §127.

From Fig. 1, $\sin (x + y) = FB = FE + EB = DC + EB$. (*a*)

$OC = \cos y$ and $BC = \sin y$. §514.

But $DC = OC \sin x = \cos y \sin x$ or $\sin x \cos y$.

and $EB = BC \cos x = \sin y \cos x$ or $\cos x \sin y$.

Substituting in (*a*), $\mathbf{\sin (x + y) = \sin x \cos y + \cos x \sin y.}$ (1)

$\cos (x + y) = OF = OD - FD = OD - EC$. (*b*)

But $OD = OC \cos x = \cos y \cos x$ or $\cos x \cos y$.

And $EC = BC \sin x = \sin y \sin x$ or $\sin x \sin y$.

Substituting in (*b*), $\cos (\mathbf{x} + \mathbf{y}) = \cos \mathbf{x} \cos \mathbf{y} - \sin \mathbf{x} \sin \mathbf{y}$. (2)

In this proof x and y and $(x + y)$ are assumed to be acute angles.

If $x + y$ is an obtuse angle, as in Fig. 2, the proof remains the same except that the sign of OF is negative.

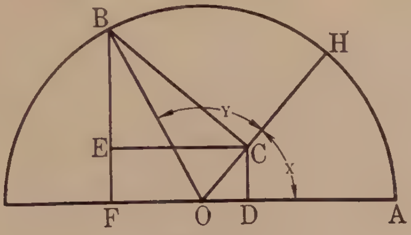


FIG. 2.

These formulas just derived are true if one of the angles is increased by 90°. Hence, if we write x' for x, where $x' = 90 + x$, we have

$\sin (x' + y) = \sin [90° + (x + y)] = \cos (x + y)$. (*c*)

$\cos (x' + y) = \cos [90° + (x + y)] = -\sin (x + y)$. (*d*)

By formula (2),

$\sin (x' + y) = \cos (x + y) = \cos x \cos y - \sin x \sin y$. (*e*)

By formula (1),

$\cos (x' + y) = -\sin (x + y) = -\sin x \cos y - \cos x \sin y$. (*f*)

Now $\cos x = \sin (90° + x) = \sin x'$ and $\sin x = -\cos (90° + x) = -\cos x'$.

Substituting these values of $\cos x$ and $\sin x$ in (*e*) and (*f*) we have,

$\sin (x' + y) = \sin x' \cos y + \cos x' \sin y$.

and $\cos (x' + y) = \cos x' \cos y - \sin x' \sin y$.

The same will be true if y is increased by 90°, or in fact if either angle is increased by 90° repeatedly. Therefore, the formulas are true for angles of any magnitude.

$$\tan (x + y) = \frac{\sin (x + y)}{\cos (x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

(Dividing (1) by (2).)

Dividing each term of the numerator and denominator by $\cos x \cos y$ we get,

$$\tan (x + y) = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

$$\therefore \tan (\mathbf{x} + \mathbf{y}) = \frac{\tan \mathbf{x} + \tan \mathbf{y}}{1 - \tan \mathbf{x} \tan \mathbf{y}}. \quad (3)$$

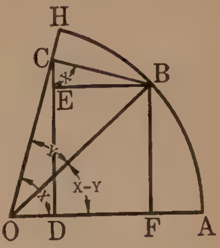
$$\cot (x + y) = \frac{\cos (x + y)}{\sin (x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}. \quad \text{Why?}$$

Dividing each term of the numerator and denominator by $\sin x \sin y$ we get,

$$\cot (\mathbf{x} + \mathbf{y}) = \frac{\cot \mathbf{x} \cot \mathbf{y} - 1}{\cot \mathbf{y} + \cot \mathbf{x}}. \quad (4)$$

535. **Functions of the Difference of Two Angles.**—In the figure let O be the center of a circle with unit radius, $\angle AOH = x$, $\angle BOH = y$, and $\angle AOB = x - y$.

Draw $BC \perp OH$, $BF \perp OA$, $CD \perp OA$, and $BE \perp CD$.



Then $\angle BCE = x$. Why?

$\sin(x - y) = BF = CD - CE$. (a)

$CD = OC \sin x = \cos y \sin x$ or $\sin x \cos y$.

$CE = BC \cos x = \sin y \cos x$ or $\cos x \sin y$.

Substituting in (a), $\mathbf{\sin(x - y) = \sin x \cos y - \cos x \sin y}$. (1)

$\cos(x - y) = OF = OD + DF = OD + EB$. (b)

$OD = OC \cos x = \cos y \cos x$ or $\cos x \cos y$.

$EB = BC \sin x = \sin y \sin x$ or $\sin x \sin y$.

Substituting in (b), $\mathbf{\cos(x - y) = \cos x \cos y + \sin x \sin y}$. (2)

By the same method as that used in (§534) these formulas can be proved true for any values of x and y.

$$\tan(x - y) = \frac{\sin(x - y)}{\cos(x - y)} = \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}.$$

Dividing each term in the numerator and denominator by $\cos x \cos y$ we get,

$$\mathbf{\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}}. \qquad (3)$$

Note.—*This formula is very important in Analytical Geometry.*

Show that $\mathbf{\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}}$. (4)

All the formulas in §535 may be derived by substituting $-y$ for y in the corresponding formulas of §534.

PROBLEMS

1. Given $\sin x = \frac{3}{5}$ and $\sin y = \frac{12}{13}$, find $\sin(x + y)$ and $\cos(x + y)$.

2. Given $\tan x = \sqrt{3}$ and $\tan y = 1$, find $\sin(x - y)$ and $\cos(x - y)$.

3. Given $\cos x = \frac{5}{13}$ and $\cos y = .8$, find $\tan(x + y)$ and $\tan(x - y)$.

4. Given $\sin x = \frac{2}{3}$ and $\sin y = .75$, find $\cot(x + y)$ and $\cot(x - y)$.

5. Given $x = 30°$ and $y = 45°$, find $\sin(x + y)$; $\cos(x + y)$; $\sin(x - y)$; and $\cos(x - y)$.

Find the numerical value of the following:

6. sin 15°. **Hint.**—*$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$.*
7. tan 15°.
8. sin 75°.
9. sin 105°.
10. cot 105°.
11. sin 15°.
12. cot 15°.
13. Find $\tan(45° + y)$ and $\tan(45° - y)$ in terms of $\tan y$.
14. Find $\cot(45° + y)$ and $\cot(45° - y)$ in terms of $\cot y$
15. Find $\sin(90° + y)$ and $\cos(90° + y)$.
16. Find $\sin(90° - y)$ and $\cos(90° - y)$.

Prove the following identities:

17. $\frac{\sin(x - y)}{\sin(x + y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$.

18. $\frac{\cos(x - y)}{\cos(x + y)} = \frac{1 + \tan x \tan y}{1 - \tan x \tan y}$.

19. $\sin(45° \pm y) = \frac{1}{2}\sqrt{2}\,(\cos y \pm \sin y)$

20. $\cot(45° \pm y) = \frac{\cot y \pm 1}{\cot y \pm 1}.$

21. $\tan(45° \pm y) = \frac{1 \pm \tan y}{1 \mp \tan y}.$

536. Functions of Double an Angle.

Given $\sin(x + y) = \sin x \cos y + \cos x \sin y$. §534–1.

Let $y = x$.

Then $\sin(x + x) = \sin 2x = \sin x \cos x + \cos x \sin x$.

$$\mathbf{\sin 2x = 2 \sin x \cos x.} \qquad (1)$$

In like manner $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - 1 + \cos^2 x$.

$$\mathbf{\cos 2x = 2 \cos^2 x - 1.} \qquad (2)$$

$$\textbf{Also } \mathbf{\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.} \qquad (3)$$

$$\mathbf{\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.} \qquad (4)$$

By the use of these formulas we can find the functions of double an angle when the functions of the angle are given.

537. Functions of Half an Angle.

$\cos^2 x + \sin^2 x = 1$. §518–1.

$\cos^2 x - \sin^2 x = \cos 2x$. §536–2.

Adding, $2 \cos^2 x = 1 + \cos 2x$.

$$\therefore \cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}.$$

Subtracting, $2 \sin^2 x = 1 - \cos 2x$.

$$\therefore \sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}.$$

Substituting θ for $2x$ and $\frac{1}{2}\theta$ for x we have,

$$\mathbf{\sin \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{2}}} \textbf{ and } \mathbf{\cos \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 + \cos\theta}{2}}.}$$

$$\textbf{Also } \mathbf{\tan \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}.} \qquad \text{§373.}$$

Hint.—*Multiply both numerator and denominator: first by* $\sqrt{1 + \cos\theta}$, *second by* $\sqrt{1 - \cos\theta}$.

$$\textbf{And } \mathbf{\cot \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 + \cos\theta}{1 - \cos\theta}} = \frac{1 + \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 - \cos\theta}.}$$

By the use of these formulas we can find the functions of half an angle when the functions of the angle are given.

538. The Sum and the Difference of Functions.

Given $\sin(x + y) = \sin x \cos y + \cos x \sin y$, (a)

$\sin(x - y) = \sin x \cos y - \cos x \sin y$, (b)

$\cos(x + y) = \cos x \cos y - \sin x \sin y$, (c)

and $\cos(x - y) = \cos x \cos y + \sin x \sin y$. (d)

Adding (a) and (b) we have,

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y, \qquad (e)$$

Subtracting (b) from (a),

$$\sin(x + y) - \sin(x - y) = 2 \cos x \sin y. \qquad (f)$$

Adding (c) and (d),

$$\cos (x + y) + \cos (x - y) = 2 \cos x \cos y. \tag{g}$$

Subtracting (d) from (c),

$$\cos (x + y) - \cos (x - y) = -2 \sin x \sin y. \tag{h}$$

Let $x + y = A$ and $x - y = B$.

Then $x = \frac{1}{2}(A + B)$ and $y = \frac{1}{2}(A - B)$.

Substituting these values in (e), (f), (g), and (h) respectively we have the following formulas:

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B). \tag{1}$$

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B). \tag{2}$$

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B). \tag{3}$$

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B). \tag{4}$$

Dividing (2) by (1) we get,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \tan \tfrac{1}{2} (A - B) \cot \tfrac{1}{2}(A + B).$$

$$\text{But } \cot \tfrac{1}{2} (A + B) = \frac{1}{\tan \frac{1}{2}(A + B)}.$$

$$\therefore \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}. \tag{5}$$

Pay particular attention to this last formula as it will be used in the solution of oblique triangles.

PROBLEMS

1. Find $\sin 3x$ in terms of $\sin x$.

Solution.—In the formula of §534–1, let $y = 2x$.

Then $\sin (x + 2x) = \sin x \cos 2x + \cos x \sin 2x$. (1)

Substituting the values of $\cos 2x$ and $\sin 2x$ in (1) we have,

$$\sin 3x = \sin x(\cos^2 x - \sin^2 x) + \cos x(2 \sin x \cos x).$$

Simplifying, $\sin 3x = \sin x(1 - \sin^2 x) - \sin^3 x + 2 \sin x(1 - \sin^2 x)$.

$$\begin{aligned} \sin 3x &= \sin x - \sin^3 x - \sin^3 x + 2 \sin x - 2 \sin^3 x. \\ &= 3 \sin x - 4 \sin^3 x. \end{aligned}$$

2. Find $\cos 3x$ in terms of $\cos x$. *Ans.* $4 \cos^3 x - 3 \cos x$

3. Find $\tan 3x$ in terms of $\tan x$. *Ans.* $\dfrac{3 \tan x - \tan^3 x.}{1 - 3 \tan^2 x}$

4. Given $\sin x = \frac{4}{5}$, find the functions of $2x$.

5. Given $\tan x = 1$, find the functions of $\frac{1}{2}x$.

6. Given $\cos \frac{1}{2} x = \frac{1}{2}$, find $\cos x$ and $\angle x$.

7. Given $\tan \frac{1}{2} x = \frac{1}{4}$, find $\cos x$.

8. Given $\cot \frac{1}{2} x = 2$, find $\cos x$.

9. Given $\sin 45° = \frac{1}{2}\sqrt{2}$, find the functions of $22° 30'$.

10. Given $\tan 60° = \sqrt{3}$, find the functions of $30°$.

11. Given $\cos 30° = \frac{1}{2}\sqrt{3}$, find the functions of $15°$.

12. If $A + B + C = 180°$, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Hint.—*$A + B = 180° - C$ and $\tan (A + B) = \tan (180° - C)$.*

13. If $A + B + C = 180°$, prove that $\sin A \cos B + \cos A \sin B = \sin C$.

14. In a right triangle show that $\cot \frac{1}{2}x = \sqrt{\dfrac{c + b}{c - b}}$.

15. Show that $\sin 50° + \sin 30° = 2 \sin 40° \cos 10°$.

Solution.—$\sin 50° + \sin 30° = 2 \sin \frac{1}{2}(50° + 30°) \cos \frac{1}{2}(50° - 30°) = 2 \sin 40° \cos 10°$.

Prove the following identities.

16. $\sin 60° + \sin 30° = \sqrt{2} \cos 15°$.

17. $\cos 60° - \cos 30° = -\sqrt{2} \sin 15°$.

18. $\sin 80° - \sin 40° = \sin 20°$.

19. $\dfrac{\cos A - \cos B}{\sin A + \sin B} = -\tan \frac{1}{2}(A - B)$.

20. $\cot 18° = \dfrac{\cos 33° + \cos 3°}{\sin 33° + \sin 3°}$.

21. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$.

22. $\dfrac{1 - \cos x}{\sin x} = \tan \frac{1}{2}x = \dfrac{\sin x}{1 + \cos x}$.

23. $\dfrac{1 + \cos x}{\sin x} = \cot \frac{1}{2}x = \dfrac{\sin x}{1 - \cos x}$.

24. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$.

25. $\dfrac{\cot x - \cot y}{\tan x - \tan y} = -\cot x \cot y$.

26. $\dfrac{\cos (x - y) - \cos (x + y)}{\sin x \sin y} = \dfrac{\cos (x + y) + \cos (x - y)}{\cos x \cos y}$.

27. $\dfrac{\sin 3x}{\cos x} + \dfrac{\cos 3x}{\sin x} = \dfrac{2}{\tan 2x}$.

28. $\tan (45° + x) - \tan (45° - x) = 2 \tan 2x$.

CHAPTER XVI

OBLIQUE TRIANGLES

539. **Formulas for the Solution of Oblique Triangles.**—The problems solved thus far depend upon the knowledge of the solution of right triangles only. It will be found, however, that many problems occur in engineering which involve the solution of oblique triangles. An oblique triangle may be solved by first dividing it into right triangles and then solving these right triangles. Such a solution is often long and offers more chances for errors than if solved by a formula into which the given parts of the triangle can be substituted directly. Before taking up the solution of oblique triangles the necessary formulas will be derived.

540. **Laws of Sines.**—In any triangle any two sides are in the same ratio as the sines of the angles opposite these sides.

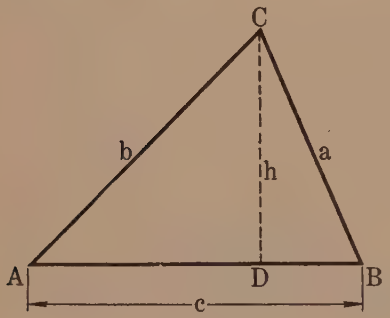


Fig. 1.

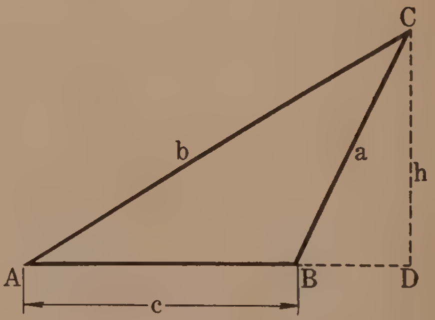


Fig. 2.

Let Fig. 1 be an acute triangle, and Fig. 2, an obtuse triangle, $\angle B$ being obtuse.

Let CD be the altitude of each triangle denoted by **h**.

In the rt. $\triangle ADC$ (Fig. 1), $h = b \sin A$.

In the rt. $\triangle DBC$, $h = a \sin B$.

$\therefore a \sin B = b \sin A$.

In the rt. $\triangle ADC$ (Fig. 2), $h = b \sin A$.

In the rt. $\triangle BDC$, $h = a \sin \angle DBC = a \sin (180° - \angle ABC)$

$= a \sin B$.

$\therefore a \sin B = b \sin A$.

Dividing both members of the equation by $b \sin B$ we have,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Hence in both figures $a : b = \sin A : \sin B$.

In the same way, by drawing altitudes from A and B we have, $b:c = \sin B:\sin C$ and $a:c = \sin A:\sin C$.

$$\therefore \frac{\mathbf{a}}{\sin \mathbf{A}} = \frac{\mathbf{b}}{\sin \mathbf{B}} = \frac{\mathbf{c}}{\sin \mathbf{C}}. \qquad \text{§ 295.}$$

541. **Law of Tangents.**—In any triangle the difference of any two sides is to their sum as the tangent of half the difference of the angles opposite these sides, is to the tangent of half the sum of the same two angles.

In any triangle $a:b = \sin A:\sin B$. §540.

By division and composition we have,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}. \qquad \text{§ 299.}$$

But
$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}. \qquad \text{§ 538–5.}$$

$$\therefore \frac{\tan \frac{1}{2}(\mathbf{A}-\mathbf{B})}{\tan \frac{1}{2}(\mathbf{A}+\mathbf{B})} = \frac{\mathbf{a}-\mathbf{b}}{\mathbf{a}+\mathbf{b}}. \qquad \text{Ax. 6.}$$

Likewise $\dfrac{\tan \frac{1}{2}(\mathbf{B}-\mathbf{C})}{\tan \frac{1}{2}(\mathbf{B}+\mathbf{C})} = \dfrac{\mathbf{b}-\mathbf{c}}{\mathbf{b}+\mathbf{c}}$, and $\dfrac{\tan \frac{1}{2}(\mathbf{C}-\mathbf{A})}{\tan \frac{1}{2}(\mathbf{C}+\mathbf{A})} = \dfrac{\mathbf{c}-\mathbf{a}}{\mathbf{c}+\mathbf{a}}$.

542. **Law of Cosines.**—In any triangle, the square of any side is equal to the sum of the squares of the other two sides diminished by twice the product of these two sides and the cosine of the included angle.

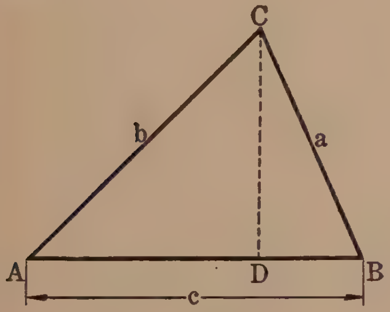


FIG. 1.

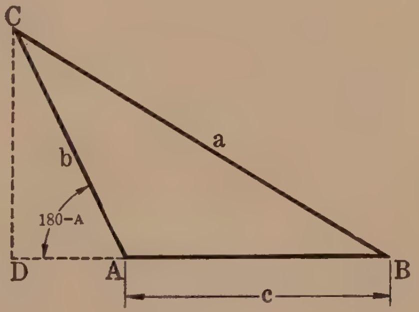


FIG. 2.

In $\triangle ABC$ (Fig. 1) $a^2 = b^2 + c^2 - 2c \times AD$. §413.

But in rt. $\triangle ADC$, $AD = b \cos A$.

Substituting for AD, $\therefore a^2 = b^2 + c^2 - 2bc \cos A$.

In $\triangle ABC$ (Fig. 2) $a^2 = b^2 + c^2 + 2c \times AD$. §414.

But in rt. $\triangle ACD$, $AD = b \cos \angle CAD = b \cos (180° - A)$
$= -\cos A$. §533–II.

$\therefore \mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2 - 2\mathbf{bc} \cos \mathbf{A}$.

Likewise $\mathbf{b}^2 = \mathbf{a}^2 + \mathbf{c}^2 - 2\mathbf{ac} \cos \mathbf{B}$, and $\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{ab} \cos \mathbf{C}$.

543. **General Relation Involving the Six Parts of a Triangle.**

$$\frac{a}{c} = \frac{\sin A}{\sin C} \;(1) \text{ and } \frac{b}{c} = \frac{\sin B}{\sin C}. \;(2) \qquad \text{§540.}$$

Subtracting (2) from (1), $\dfrac{a-b}{c} = \dfrac{\sin A - \sin B}{\sin C}$. (3) *Ax.* 2.

But $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$. (4) §538–2. Why?

Also $\frac{1}{2}(A + B)$ and $\frac{1}{2} C$ are complementary.

$$\therefore \cos \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2}C. \qquad (5) \quad \text{§517.}$$

Substituting from (4) and (5) in (3),

$$\frac{a - b}{c} = \frac{2 \sin \frac{1}{2}C \sin \frac{1}{2}(A - B)}{\sin C}. \qquad (6)$$

Multiplying both numerator and denominator of the right hand member of (6) by $\cos \frac{1}{2}C$ we have,

$$\frac{a - b}{c} = \frac{2 \sin \frac{1}{2}C \cos \frac{1}{2} C \sin \frac{1}{2} (A - B)}{\sin C \cos \frac{1}{2} C} \qquad (7)$$

But $2 \sin \frac{1}{2} C \cos \frac{1}{2} C = \sin C$. (8) §536–1

Substituting from (8) in (7)

$$\frac{a - b}{c} = \frac{\sin C \sin \frac{1}{2}(A - B)}{\sin C \cos \frac{1}{2} C} = \frac{\sin \frac{1}{2} (A - B)}{\cos \frac{1}{2} C}. \qquad (9)$$

Clearing of fractions, $(a - b) \cos \frac{1}{2} C = c \sin \frac{1}{2} (A - B)$. (10)

Relation (10) will be used for checking the solution of oblique triangles. It is a reliable check because it involves the six parts of the triangle.

544. Solution of Oblique Triangles.—The three formulas derived above will now be used in solving oblique triangles. We must remember that $A + B + C = 180°$. Read again §§521 and 523.

These solutions may be classified as follows:

I. Two angles and a side.
II. Two sides and the angle opposite one of them
III. Two sides and the included angle.
IV. Three sides.

545. Case I.—Two Angles and a Side.

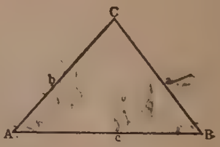


Illustration.

Given $A = 48° 25'$, $B = 52° 48'$, and $a = 384.52$.

To find C, b, and c.

Solution.

1. $C = 180 - (A + B) = 78° 47'$.

2. $\frac{a}{b} = \frac{\sin A}{\sin B}$. $\therefore b = \frac{a \sin B}{\sin A}$. 3. $\frac{a}{c} = \frac{\sin A}{\sin C}$. $\therefore c = \frac{a \sin C}{\sin A}$.

$\log a = 2.58492$	$\log a = 2.58492$
$\log \sin B = 9.90120$	$\log \sin C = 9.99162$
$\text{colog} \sin A = 0.12610$	$\text{colog} \sin A = 0.12610$
$\log b = 2.61222$	$\log c = 2.70264.$
$\therefore b = 409.47.$	$\therefore c = 504.24.$

Check

$$(b - a) \cos \tfrac{1}{2} C = c \sin \tfrac{1}{2}(B - A). \qquad \text{§543–10}$$

Since $B > A$, B and A as well as b and a must be interchanged in the formula.

$$\begin{aligned} a &= 384.52 \\ b &= 409.47 \\ c &= 504.24 \\ \therefore b - a &= 24.95 \\ \tfrac{1}{2}C &= 39°23'30''. \\ \log(b-a) &= 1.39707 \\ \log\cos\tfrac{1}{2}C &= 9.88808 \end{aligned}$$

$$\begin{aligned} A &= 48°\,25' \\ B &= 52°\,48' \\ C &= 78°\,47' \\ B - A &= 4°\,23' \\ \tfrac{1}{2}(B-A) &= 2°\,11'\,30''. \\ \log c &= 2.70264 \\ \log\sin\tfrac{1}{2}(B-A) &= 8.58254 \end{aligned}$$

Adding logs, 1.28515. Adding logs, 1.28518.

The logs check within 3 in the last place, which is near enough.

PROBLEMS

1. Given $a = 126.32$, $A = 65°\,35'$, $B = 73°\,18'$, find C, b, and c.
Ans. $b = 132.88$, $c = 91.226$, $C = 41°\,7'$.

2. Given $B = 50°\,37'$, $C = 68°\,11'$, $c = 2.8364$, find A, a, and b.

3. Given $A = 26°\,10'\,45''$, $C = 44°\,35'\,12''$, $b = .6708$, find a, c, and B.
Ans. $a = .31344$, $c = .49872$.

4. Given $B = 82°\,43'\,2''$, $C = 55°\,37'\,48''$, $a = 38456$, find A, b, and c.

5. Given $a = 6412$, $A = 70°\,55'$, $C = 52°\,9'$, find b, c, and B.
Ans. $B = 56°\,56'$, $b = 5685.9$, $c = 5357.5$.

6. Given $c = 38.459$, $B = 105°\,26'$, $C = 60°\,12'$, find a, b, and A.

7. Given $b = .04256$, $A = 38°\,46'\,15''$, $C = 33°\,26'\,40''$, find a, c, and B.

8. A ship K can be seen from two points A and B on the shore. The distance AB is 4890 feet, and angles KAB and KBA are $62°\,37'$ and $73°\,45'$ respectively. Find the distance from B to the ship.

9. An airplane is directly over a straight level road-way and between two observers in the road. The observers are 3540 yards apart and the angles of elevation of the airplane as measured simultaneously by the observers are $46°\,28'\,15''$ and $58°\,45'\,30''$, respectively. Find the distance from each observer to the airplane and the distance from the airplane to the roadway. Draw figure to approximate scale.

10. The longer side of a parallelogram is 10.32 feet and the angles between this side and the diagonals are $47°\,32'$ and $30°\,45'$. Find the diagonals of the parallelogram.

546. Case II. Two Sides and the Angle Opposite One of Them.—In this case we may have one or two solutions, or it may be possible that there is no solution.

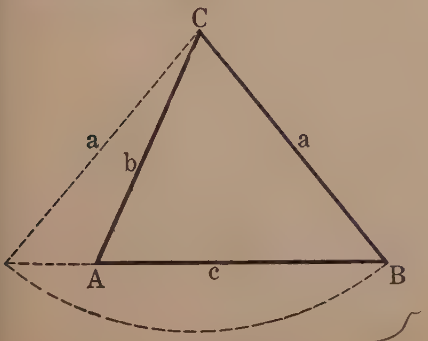


FIG. 1.

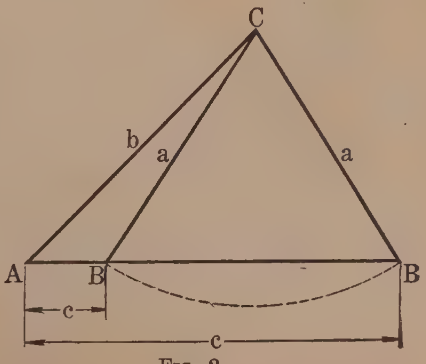


FIG. 2.

(a) There will be one solution:

1. When A is acute and $a > b$ (Fig. 1).
2. When A is acute and $a = b \sin A$ (rt. △).
3. When A is acute and $a = b$ (Isos. △).
4. When A is obtuse and $a > b$. Draw a figure.

(b) There will be two solutions when A is acute and $a < b$, but $a > b \sin A$ (Fig. 2).

(c) There will be no solution:

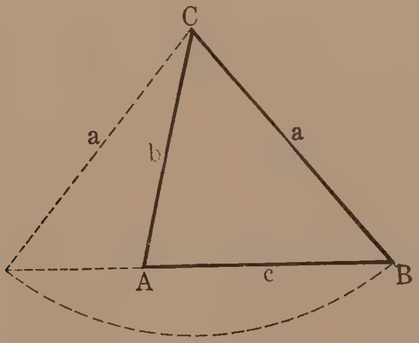
1. When A is acute and $a < b \sin A$.
2. When A is obtuse and $a < b$ or $a = b$.

Illustrate the last two cases by drawing figures.

Illustrations.

1. Given $a = 25.35$, $b = 20.36$, and $A = 68° \; 28' \; 45''$.

Find c, B, and C.



The side opposite the acute angle is greater than the adjacent side, hence only one solution.

Solution.—From the law of sines we have,

$$1. \quad \frac{a}{\sin A} = \frac{b}{\sin B}. \qquad \therefore \sin B = \frac{b \sin A}{a}.$$

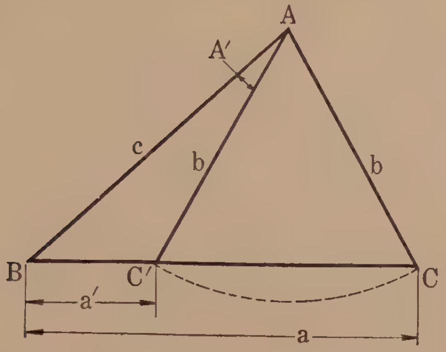
$$\begin{aligned} \log b &= 1.30878 \\ \log \sin A &= 9.96862 \\ \operatorname{colog} a &= 8.59602 - 10 \\ \hline \log \sin B &= 9.87342 \\ B &= 48° \, 20' \, 44''. \end{aligned}$$

$$2. \quad C = 180 - (A + B) = 63° \, 10' \, 31''.$$

$$3. \quad \frac{a}{\sin A} = \frac{c}{\sin C}, \quad \therefore c = \frac{a \sin C}{\sin A}.$$

$$\begin{aligned} \log a &= 1.40398 \\ \log \sin C &= 9.95056 \\ \operatorname{colog} \sin A &= 0.03138 \\ \hline \log c &= 1.38592 \\ c &= 24.318. \end{aligned}$$

2. Given $b = 8.425$, $c = 9.582$, and $B = 52° 38'$. Find a, a', A, A', C, and C'.



Since the side **b**, opposite the acute angle **B** is less than the adjacent side **c**, and is greater than $c \sin B$, there will be two solutions.

Solutions.

1. $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, $\therefore \sin C = \dfrac{c \sin B}{b}$.

2. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. $\therefore a = \dfrac{b \sin A}{\sin B}$ and $a' = \dfrac{b \sin A'}{\sin B}$.

$$\begin{aligned} \log c &= 0.98146 \\ \log \sin B &= 9.90024 \\ \operatorname{colog} b &= 9.07443 \\ \hline \log \sin C &= 9.95613 \\ C &= 64° 40' 40''. \end{aligned}$$

$$\begin{aligned} \log b &= 0.92557 \\ \log \sin A &= 9.94867 \\ \operatorname{colog} \sin B &= 0.09976 \\ \hline \log a &= 0.97400 \\ a &= 9.419. \end{aligned}$$

3. $A = 180° - (B + C) = 62° 41' 20''$.
$C' = 180° - C = 115° 19' 20''$.
$A' = 180° - (B + C') = 12° 2' 40''$.

Results

$$\begin{aligned} a &= 9.419 \\ a' &= 2.212 \\ A &= 62° 41' 20'' \\ A' &= 12° 2' 40'' \\ C &= 64° 40' 40'' \\ C' &= 115° 19' 20''. \end{aligned}$$

$$\begin{aligned} \log b &= 0.92557 \\ \log \sin A' &= 9.31946 \\ \operatorname{colog} \sin B &= 0.09976 \\ \hline \log a' &= 0.34479 \\ a' &= 2.212. \end{aligned}$$

PROBLEMS

Draw a figure approximately to scale for each problem before solving to determine how many solutions if any.

1. Given $a = 56$, $b = 45$, $A = 45° 26'$.
2. Given $b = 384.6$, $c = 562.3$, $C = 38° 42'$.
3. Given $a = 64.385$, $c = 85.605$, $A = 64° 25'$.
4. Given $a = 64.385$, $c = 85.605$, $A = 28° 40'$.
5. Given $a = 64.385$, $c = 85.605$, $A = 48° 46' 33''$.
6. Given $a = 38.642$, $b = 74.891$, $B = 102° 28'$.

7. Given $b = 7.462$, $c = 6.385$, $C = 42° 32' 15''$.
8. Given $a = 35.68$, $b = 23.46$, $B = 31° 22'$.
9. Given $a = 939.92$, $b = 919.75$, $A = 121° 36'$.
10. Given $b = 12.111$, $c = 8.268$, $C = 31° 41' 45''$.
11. Given $b = 24.45$, $c = 39.38$, $B = 58° 22'$.

547. Case III.—Two Sides and the Included Angle.

Illustration.

Given $b = 45.256$, $c = 32.485$, and $A = 58° 25' 20''$. Find a, B, and C.

Solution.—From the law of tangents we have,

1. $\tan \frac{1}{2}(B - C) = \frac{b - c}{b + c} \tan \frac{1}{2}(B + C)$.

$$B + C = 180° - 58° 25' 20'' = 121° 34' 40''.$$

77 ' 9 22

1 6

$$\tfrac{1}{2}(B + C) = 60° 47' 20''$$
$$b = 45.256$$
$$c = 32.485$$

$$b + c = 77.741$$
$$b - c = 12.771$$

$$\log (b - c) = 1.10622$$
$$\text{colog } (b + c) = 8.10936$$
$$\log \tan \tfrac{1}{2}(B + C) = 0.25248$$

$$\text{logtan } \tfrac{1}{2}(B - C) = 9.46806$$
$$\tfrac{1}{2}(B - C) = 16° 22' 22''$$
$$\tfrac{1}{2}(B + C) = 60° 47' 20''$$

02

Adding, $B = 77° 9' 42''$.

Subtracting, $C = 44° 24' 58''$.

2. $\frac{a}{\sin A} = \frac{b}{\sin B}$.

$$\therefore a = \frac{b \sin A}{\sin B}.$$

$$\log b = 1.65567$$
$$\log \sin A = 9.93041$$
$$\text{colog } \sin B = 0.01100$$

$$\log a = 1.59708$$
$$a = 39.544$$

Results

$a = 39.544'$

$B = 77° 9' 42''$

$C = 44° 24' 58''$.

PROBLEMS

1. Given $a = 25$, $b = 30$, and $C = 35° 28'$; find c, A and B.
2. Given $b = 384.2$, $c = 425.8$, and $A = 57° 25'$; find a, B and C.
3. Given $a = 62.874$, $c = 50.385$ and $B = 102° 38' 30''$; find b, A and C.
4. Given $b = .8647$, $c = .7642$, and $A = 80° 52' 45''$; find a, B and C.
5. Given $a = .0284$, $c = .0236$, and $B = 28° 56' 40''$, find b, A and C.
6. Given $a = 3.7642$, $b = 2 9686$, and $C = 98° 48' 45''$; find c, A and B.
7. Given $b = 89721$, $c = 7023.4$, and $A = 48° 48' 25''$; find a, B and C.
8. It is desired to find the distance between two points A and C. B was taken so that $AB = 2874.5$ feet, $BC = 3142.4$ feet, and $\angle ABC = 63° 20' 20''$. Find AC. Draw a figure.
9. The point P is acted on by a force PA of 20 pounds and a force PB of 25 pounds. The angle between the lines of direction of the two forces is 108° 28'. What

will be the resultant force and what angle will it make with each of the original forces? (Use the principle of the parallelogram of forces.)

10. Two trains running at the rates of 25 and 40 miles per hour respectively start at the same time from the same point and move along straight level railroads, which intersect at an angle of 64° 40′. How far apart are they at the end of 72 minutes?

548. Case IV.—Three Sides.

Given the three sides of a triangle; a, b, and c.

From the law of cosines we have $a^2 = b^2 + c^2 - 2bc \cos A$.

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}. \qquad (1) \quad \S 542.$$

To adapt this formula to logarithmic work we proceed as follows:

(a) Subtracting both sides of equation (1) from 1 we have,

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc},$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc},$$

Factoring, $$= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc}. \qquad (2)$$

To simplify, let $a + b + c = 2s$. Then $b + c - a = 2(s - a)$, $a - b + c = 2(s - b)$, and $a + b - c = 2(s - c)$. §433.

Substituting in (2), $$1 - \cos A = \frac{2(s - c)\cancel{2}(s - b)}{\cancel{2}bc} = \frac{2(s - c)(s - b)}{bc}$$

(b) Adding 1 to both sides of equation (1) we have,

$$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc}.$$

Factoring,

$$= \frac{(b + c + a)(b + c - a)}{2bc}.$$

Substituting as above,

$$1 + \cos A = \frac{2s\cdot\cancel{2}(s - a)}{\cancel{2}\,bc} = \frac{2s(s - a)}{bc}.$$

But $1 - \cos A = 2 \sin^2 \frac{1}{2}A$, and $1 + \cos A = 2 \cos^2 \frac{1}{2}A$. §537.

$$\therefore\ 2 \sin^2 \tfrac{1}{2}A = \frac{2(s - b)(s - c)}{bc}, \text{ and } 2 \cos^2 \tfrac{1}{2}A = \frac{2s(s - a)}{bc}. \qquad Ax.\ 2.$$

$$\therefore\ \sin^2 \tfrac{1}{2}A = \frac{(s - b)(s - c)}{bc}, \ (3) \text{ and } \cos^2 \tfrac{1}{2}A = \frac{s(s - a)}{bc}. \ (4) \qquad Ax.\ 4.$$

Dividing (3) by (4),

$$\frac{\sin^2 \frac{1}{2}A}{\cos^2 \frac{1}{2}A} = \tan^2 \tfrac{1}{2}A = \frac{(s - b)(s - c)}{s(s - a)}$$

$$\text{or } \tan \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}. \qquad (5)$$

$$\text{Similarly, } \tan \tfrac{1}{2}B = \sqrt{\frac{(s - a)(s - c)}{s(s - b)}},$$

$$\text{and } \tan \tfrac{1}{2}C = \sqrt{\frac{(s - a)(s - b)}{s(s - c)}}.$$

Note the arrangement of a, b, and c in the formulas.

Multiplying both numerator and denominator of the radical in (5) by $\sqrt{s-a}$ we have,

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}} \quad\text{or}\quad \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\text{But } \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = r. \qquad \S 446.$$

$$\therefore \tan \tfrac{1}{2}A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2}B = \frac{r}{s-b}, \quad\text{and}\quad \tan \tfrac{1}{2}C = \frac{r}{s-c}. \qquad (6)$$

Illustration.

Given $a = 21.42$, $b = 24.68$ and $c = 30.82$. Find A, B, and C.

Solution.

$$\begin{aligned} a &= 21.42 \\ b &= 24.68 \\ c &= \underline{30.82} \\ 2s &= 76.92 \\ s &= 38.46 \\ s-a &= 17.04 \\ s-b &= 13.78 \\ s-c &= 7.64 \end{aligned}$$

$$\begin{aligned} \log(s-a) &= 1.23147 && (1) \\ \log(s-b) &= 1.13925 && (2) \\ \log(s-c) &= 0.88309 && (3) \\ \operatorname{colog} s &= \underline{8.41499} && (4) \\ \log r^2 &= 1.66880 && (5) \\ \log r &= .83440 && (6) \end{aligned}$$

Subtracting (1), (2), and (3) in turn from (6) we have,

$$\begin{aligned} \log \tan \tfrac{1}{2}A &= 9.60293, & \therefore \tfrac{1}{2}A &= 21° \, 50' \, 28'' \\ \log \tan \tfrac{1}{2}B &= 9.69515, & \therefore \tfrac{1}{2}B &= 26° \, 21' \, 50.5'' \\ \text{and } \log \tan \tfrac{1}{2}C &= 9.95131. & \therefore \tfrac{1}{2}C &= \underline{41° \, 47' \, 41.5''} \\ & & A &= 43° \, 40' \, 56'' \\ & & B &= 52° \, 43' \, 41'' \\ & & C &= \underline{83° \, 35' \, 23''} \\ & & \text{Check } A + B + C &= 180° \, 0' \, 0'' \end{aligned}$$

Note.—*It sometimes happens, even if there are no mistakes in the work, that the sum of the angles will not be exactly equal to 180°. In this case the difference should be divided among the angles according to the probable amount of error for each angle.*

PROBLEMS

Find the angles of the following triangles having given the three sides

1. $a = 16$, $b = 25$, $c = 35$.
2. $a = 1.428$, $b = 1.943$, $c = 1.647$.
3. $a = 82.63$, $b = 78.41$, $c = 70.44$.
4. $a = 384.25$, $b = 338.42$, $c = 401.37$.
5. $a = .2864$, $b = .2472$, $c = .2248$.
6. $a = .03785$, $b = .03848$, $c = .03168$.
7. $a = 45623$, $b = 38472$, $c = 35721$.
8. $a = 62.423$, $b = 72.463$, $c = 82.478$.

9. A surveyor selected three points A, B, and C. The distances between the points were found to be, $AB = 324.6$ feet, $BC = 432.8$ feet, and $CA = 386.4$ feet. Find each angle of the triangle formed.

10. If the sides of a triangle are 11, 13, and 15, find the sine of the largest angle.

549. Area of Oblique Triangles.

1. When two sides and the included angle are given.

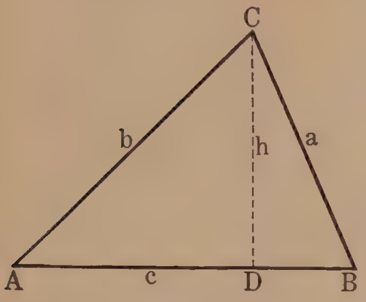


FIG. 1.

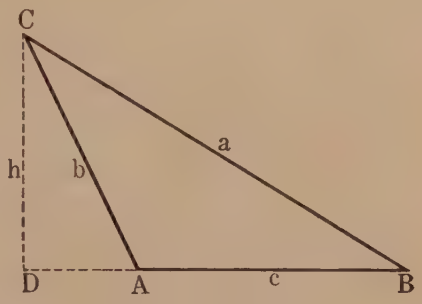


FIG. 2.

In the triangle ABC (either figure) A = Area. $\therefore A = \frac{1}{2}ch$.
But $h = a \sin B = b \sin A$.

$$\therefore \mathbf{A} = \tfrac{1}{2}\mathbf{ac} \sin \mathbf{B} \text{ or } \tfrac{1}{2}\mathbf{bc} \sin \mathbf{A}.$$

Similarly $$\mathbf{A} = \tfrac{1}{2}\mathbf{ab} \sin \mathbf{C}.$$

2. When two angles and a side are given.

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \quad \therefore c = \frac{a \sin C}{\sin A}.$$

Substituting for c in the formulas of (1) we have,

$$\mathbf{A} = \frac{\mathbf{a}^2 \sin \mathbf{B} \sin \mathbf{C}}{2 \sin \mathbf{A}}.$$

3. When the three sides are given, use the formula

$$\mathbf{A} = \sqrt{\mathbf{s}(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}. \qquad \S 444.$$

PROBLEMS

Find the area of a triangle having given:

1. $a = 48.62$, $b = 39.46$, and $C = 68° 24' 10''$.
2. $a = 384.85$, $c = 334.63$, and $B = 102° 25' 30''$.
3. $b = 6.754$, $A = 96° 22'$, and $B = 70°, 20' 20''$.
4. $a = 5.6892$, $b = 8.3106$, and $c = 6.9421$.
5. $c = 3864.2$, $A = 38° 40' 45''$, and $B = 68° 20' 20''$.
6. $b = 200$, $c = 150$, and $A = 72° 18' 30''$.
7. $a = 672.35$, $A = 48° 50' 50''$, and $B = 78° 20' 30''$.
8. Prove that the area of any quadrilateral is equal to one half the product of its diagonals and the sine of their included angle.
9. The diagonals of a quadrilateral are 14.92 and 19.36, intersecting at an angle of 40° 20′. Find the area of the quadrilateral.
10. Given A, B, C, and D the vertices of a field. The following measurements were taken: $AB = 106.4$ rods, $BC = 88.3$ rods, $CD = 92.7$ rods, $DA = 60.5$ rods, and $AC = 110.8$ rods. Find the number of acres in the field.

APPLICATIONS

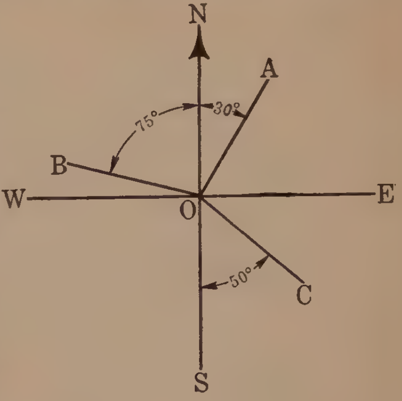
550. **Bearing.**—The bearing of a line is the acute angle which the line makes with a north and south line.

Illustrations.—In the figure let O be the point of observation. The bearing of OA is north, 30° east, and is written N. 30° E.

The bearing of OB is north, 75° west, and is written N. 75° W. The bearing of OS is due south.

The bearing of OC is south, 50° east, and is written S. 50° E.

Note.—*A bearing is always read from the north or south point and never from the east or west.*



PROBLEMS

1. AB is a straight line extending due north from A a distance of 537 feet. At A a point P bears N 40° E. At B the same point P bears N. 70° E. Find AP and BP. Draw a figure and check.

Ans. $AP = 1009.23$; $BP = 690.35$.

2. MN is a straight line extending N. 30° E from M a distance of 500 feet. At M a point P bears N 50° E. and at N the same point P bears N 75° E. Find MP and NP.

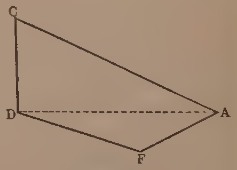
3. AB is a base line 825 feet long, extending N. 30° 30′ W from A. At A a point P bears S 75° 30′ W and at B it bears S 40 W. Find the distance from A and B to P.

4. It is required to find the distance from a point B to a point A across a stream. A base line BC was laid off 117.83 feet. Angles ABC and BCA were measured and found to be 76° 48′ 30″ and 53° 31′ respectively. Find AB. This is the same distance found in problem 57, §525.

5. To locate a buoy C in a river, a base line AB was laid off 200 feet long, angle CAB is 56° 27′ and angle ABC is 40° 20′ 30″. Find the distances of the buoy from A and B and also from the line AB.

6. A ship lies in a river 428 yards wide. From two points of the same elevation on opposite banks of the river and in a straight line with the mast, the angles of elevation to the top of the mast were observed to be 11° 48′ and 13° 25′ respectively. Find the height of the mast above the points of observation.

7. To find the height of an inaccessible point C, the following data were taken: The angle of elevation $DAC = 24° 22′$. The horizontal angles DAF and AFD are 78° 34′ and 73° 13′ respectively. $AF = 135$ feet. Find CD.



PROB. 7.

8. From the ridge of a mountain, the angles of depression to the ends of a tunnel at the base are found to be 55° 30′ and 68° 10′ respectively. The distances measured down the slope to the ends of the tunnel are 3475 and 2896 feet. The point of observation and the ends of the tunnel must be in the same vertical plane. Find the length of the tunnel.

9. A tree standing on the side of a hill which rises at an angle of 15° 45′ breaks off 28 feet above the ground. The top strikes down the slope 32 feet from the foot

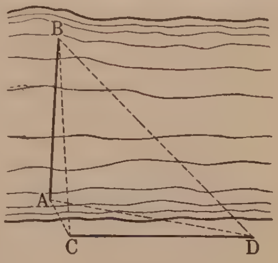
of the tree. How high was the tree? (*a*) If the distance was measured on the slope. (*b*) If the distance was measured horizontally.

10. Three forces in the same plane acting on a body produce equilibrium. The least force is 6 pounds, and the greatest is 10 pounds. The angle between the 6 pound force and the unknown force is 79° 43′. Find the third force.

11. Two forces of 16.2 pounds and 12.5 pounds making an angle of 108° 20′ act on the same body at the same time. Find the resultant and the angle it makes with each force.

12. To find the main span AB of a bridge, a base line CD was taken 150 feet long. Angles were measured as follows: $ACB = 25°\ 6'$, $BCD = 95°\ 10'$, $BDA = 35°\ 0'$, and $ADC = 10°\ 14'$. Find AB.

Ans. 136.15 feet.



Prob. 12.

13. It is required to find the distance between two points A and B on opposite sides of a building. Points C and F are chosen so that sights may be taken on A and B. The field book showed the following data:

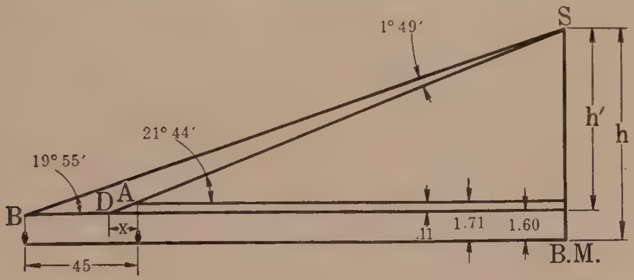
$AC = 310.05$ ft., $CB = 294.95$ ft., $\angle C = 98°\ 46'$.
$AF = 311.78$ ft., $FB = 419.42$ ft., $\angle F = 76°\ 22'\ 30''$.

Compute AB from both sets of data and compare results. *Ans.* 459.36 ft.

14. To find the elevation of the top of the statue on Memorial Hall, Fairmount Park, Philadelphia, the following data were taken:

Station	Vert ∠	Rod	Distance
A	21° 44′	1.71′	*AB*
B	19° 55′	1.60′	45′

Stations A and B were taken in the same straight line with the statue. The rod was held on a bench mark (BM) whose elevation above sea level is known to be 113.87 feet. Study carefully the accompanying figure and solution.



Note.—*In getting data for measuring the height of objects it is not possible to set up the transit at the same elevation any two successive times. Then, too, the stations on which the instrument is set will not all be in the same horizontal plane as the point above which the elevation is desired. It is, therefore, necessary to take a rod reading at each set up to determine the difference in elevation between the point on which the rod is held and the line of sight of the transit.*

The following solution will illustrate this point.

Solution. $x = .11 \cot A$. $\therefore BD = 45 - .11 \cot A$.

$$\frac{BS}{\sin 21°\,44'} = \frac{(45 - .11 \cot 21°\,44')}{\sin 1°\,49'}. \qquad \S 540.$$

$$\therefore BS = \frac{(45 - .11 \cot 21°\,44') \sin 21°\,44'}{\sin 1°\,49'}.$$

$h' = BS \sin 19° 55'$. Why?

$$\therefore h' = \frac{(45 - .11 \cot 21° 44') \sin 21° 44' \sin 19° 55'}{\sin 1° 49'} = 177.96 \text{ ft.}$$

$h = h' + 1.60$ ft. or 179.56 ft.

Height above sea level is 179.56 + 113.87 = 293.43 ft.

15. To find the height of Smith Memorial, Fairmount Park, Philadelphia, the following data was taken, the rod being held at the base of the monument. Three stations were taken so the height may be computed from three sets of data. *A*, *B* and *C* are in the same straight line with the monument.

Station	Vert ∠	Rod	Distance
A	40° 18′	4.73′	*AB* = 80.45′
B	29° 30′	5.33′	*BC* = 154.10′
C	19° 26′	2.93′	

Draw a figure and solve by at least two pairs of triangles for a check.

Ans. 143.33 ft.

16. To find the distance between two inaccessible high points *P* and *Q*, a base line *XY* was laid off 150 ft. long. The following angles were measured. ∠ *XYP* = 40° 14′, ∠ *XYQ* = 139° 20′, ∠ *QXY* = 25°, and ∠ *PXY* = 119° 12′. Draw a figure and solve for *PQ*.

17. To find the distance between two points *A* and *B* on opposite sides of a pond, a point *C* is chosen from which both *A* and *B* are visible. The following measurements are then taken. ∠ *C* = 77° 41′ 30″, *CA* = 88.50′, and *CB* = 79.70′.

18. To find the distance between two inaccessible points *C* and *D*, a base line *AB* was laid off 80 feet long.

The following angles were measured. ∠ *CAD* = 47° 7′ 30″, ∠*CAB* = 78° 43′, ∠ *ABC* = 38° 58′, and ∠ *ABD* = 116° 39′ 30″. Draw a figure and solve for *CD*.

19. Find the height of the statue on Memorial Hall, Fairmount Park, Philadelphia from the following data.

Station	Vert ∠ to foot	Vert ∠ to top	Rod	Distance
				AB
A	21° 49′	23° 50′	1.32′	
B	17° 42′	19° 26′	3.08′	92.50′

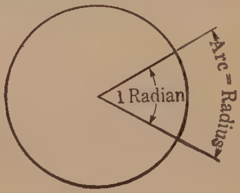
Stations *A* and *B* were taken in the same straight line with the statue and the rod readings were taken on the same bench mark.

CHAPTER XVII

SUPPLEMENTARY TOPICS

CIRCULAR MEASURE

551. **Radian.**—A radian is an angle such that if the vertex is at the center of a circle, the sides intercept an arc equal in length to the radius of the circle. The number of radians in an angle, therefore, is given by the number of times the radius is contained in the intercepted arc. This relation may be written in the form of an equation. Thus,



$$\textbf{Number of radians in an angle} = \frac{\textbf{arc}}{\textbf{radius}}.$$

$$\therefore \textbf{arc} = \textbf{number of radians} \times \textbf{radius}.$$

The symbol for radians is r. Instead of writing 2 radians, it may be written 2^r.

552. **Relations between Degrees and Radians.**—The circumference of a circle contains 360° and its length is 2π times the radius.

The radius is contained in the circumference 2π times. Hence we have,

$$2\pi^r = 360°.$$

$$\pi^r = 180°.$$

$$\therefore 1^r = \frac{180°}{\pi} = 57.3° \text{ or } 57°\ 17'\ 45'' \text{ (nearly)}.$$

$$\text{Also } 1° = .01745^r.$$

Therefore,

1. *To express any given number of radians in degrees, multiply by* $\frac{180}{\pi}$.

$$\textbf{Number of radians} \times \frac{180}{\pi} = \textbf{number of degrees.}$$

2. *To express any given number of degrees in radians, multiply by* $\frac{\pi}{180}$.

$$\textbf{Number of degrees} \times \frac{\pi}{180} = \textbf{number of radians.}$$

Note.—*The sine and tangent of angles of 2° and less differ very little from the number of radians in the angle. For this reason right triangles with one very small angle may be solved, with very little error, by the use of radians.*

PROBLEMS

1. Express the following angles, given in radians, to equivalent angles in degrees.

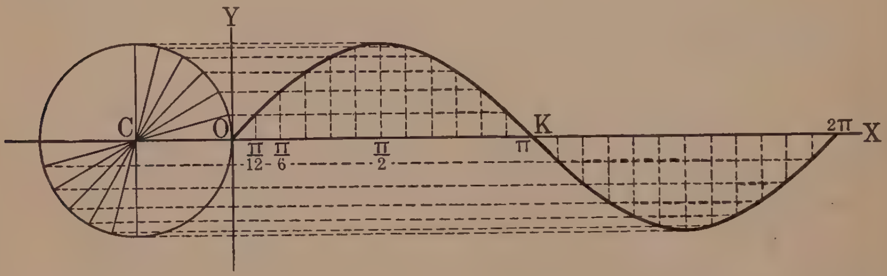
$$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{3\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{5}, \frac{\pi}{8}.$$

2. Express the following angles in radians.

$$90°, 60°, 30°, 45°, 135°, 75°, 15°, 22\tfrac{1}{2}°, 210°.$$

553. **Graphs of the Trigonometric Functions.**—The representation of the trigonometric functions in §514 furnishes a simple method for plotting the graphs of these functions.

1. **The Sine Curve.**—Given $y = \sin x$. Then $y = 0$ when $x = 0, \pi, 2\pi$, or in general $n\pi$. This shows that the curve passes through the origin and crosses the x-axis at intervals of π.



To plot the sine curve, let C be a circle with unit radius. Draw the x-axis and the y-axis as shown in the figure. From the origin lay off OK equal to the length of the semi-circumference of the circle, or π. For convenience divide OK into twelve equal parts by the method of §213 and draw vertical lines through these points of division. Divide the first and third quadrants of the circle into six equal angles, each being 15° or $\frac{\pi}{12}$ radians. Now the vertical lines from the ends of the radii to the diameter represent the sines of the respective angles. Through the outer extremities of the radii draw lines parallel to the x-axis. By drawing a smooth curve through the points of intersection of the corresponding pairs of vertical and horizontal lines the sine curve is traced.

Note.—*Since* $\sin 105° = \sin 75°$, $\sin 120° = \sin 60°$ *etc. and* $\sin 195° = \sin 345°$, $\sin 210° = \sin 330°$ *etc., if the horizontal lines are drawn long enough, as many arches of the sine curve may be plotted as desired.*

2. **The Cosine Curve.**—To plot the cosine curve, lay off the successive values of the cosine on the corresponding vertical lines of division. Connect the points thus determined by a smooth curve.

Curves of any of the trigonometric functions may be drawn by following the same method.

INVERSE FUNCTIONS

554. **Inverse Trigonometric Functions.**—If a function of an angle is more important in a problem than the value of the angle in degrees or

radians, the inverse notation is used. Thus, instead of $\tan y = 1$ we have, **y** is the angle whose tangent is 1. It is written **$y = \tan^{-1} 1$** or **$y = \text{arc} \tan 1$.** In the expression $\tan^{-1}x$, -1 is not to be considered as an exponent. It is here used only to express the inverse function.

If the angle is measured in radians, then the expression, **$y = \text{arc} \tan 1$,** means that **y** is the number of radians in the angle or arc whose tangent is 1. It must be remembered that **y** is an angle.

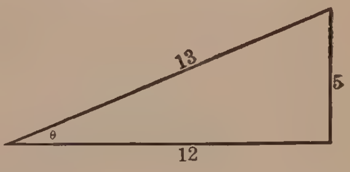
In many problems, it is not as necessary to know the value of an angle as it is to know a function of that angle. When the grade of a street is given, we have given the tangent of the angle which the street line makes with a horizontal line. It is the tangent of the angle and not the angle itself which is needed to solve problems 54, 55, and 56, §525. The same reasoning applies to the problems on "the coefficient of friction" in the same article.

In analytical geometry, the tangent of the angle which a line makes with the positive direction of the x-axis is more important than the value of the angle. Hence the inverse notation is used.

555. **Values of Direct and Inverse Trigonometric Functions.**—In the direct function, $y = \sin 60°$, y has only one value, $\frac{\sqrt{3}}{2}$. If $y = \sin^{-1} \frac{\sqrt{3}}{2}$, y can have any number of values, as 60°, 120°, 420°, 480°, etc. Thus in solving problems in which inverse functions are given certain limitations are usually put upon the number of values which the angle may have in that particular problem. In most cases the smallest angle is the angle used. Problems involving inverse trigonometric functions are most easily solved by first constructing the acute angle, one of whose functions is given.

Illustrations.—1. Given $\theta = \sin^{-1} \frac{5}{13}$. Find all the functions of θ.

Solution. $\theta = \sin^{-1} \frac{5}{13}$ may be written $\sin \theta = \frac{5}{13}$.



Hence from the figure,

$\sin \theta = \frac{5}{13}$, $\csc \theta = \frac{13}{5}$,

$\cos \theta = \frac{12}{13}$, $\sec \theta = \frac{13}{12}$,

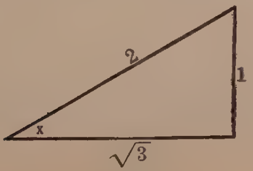
$\tan \theta = \frac{5}{12}$, $\cot \theta = \frac{12}{5}$.

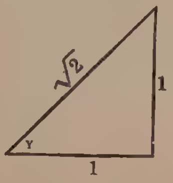
2. Find the value of $\sin (\sin^{-1} \frac{1}{2} + \tan^{-1} 1)$.

Solution.—Let $x = \sin^{-1} \frac{1}{2}$ and $y = \tan^{-1} 1$.

Then, $\sin (\sin^{-1} \frac{1}{2} + \tan^{-1} 1) = \sin (\sin^{-1} \frac{1}{2}) \cos (\tan^{-1} 1) +$
$\cos (\sin^{-1} \frac{1}{2}) \sin (\tan^{-1} 1)$. (1) §534–1

Construct angles x and y as in illustration 1.





LOGARITHMS

Now substitute in (1) the values of the functions from the figures. Hence,

$$\sin\left(\sin^{-1}\frac{1}{2}+\tan^{-1}1\right)=\frac{1}{2}\cdot\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}=\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}}=$$
$$\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{1}{4}\left(\sqrt{2}+\sqrt{6}\right).$$

PROBLEMS

Find the smallest value of each of the following angles.
(*a*) in degrees, (*b*) in radians.

1. $\sin^{-1}\frac{1}{2}$; $\tan^{-1}1$; $\cos^{-1}\frac{1}{2}$; $\sin^{-1}\frac{1}{2}\sqrt{2}$; $\cos^{-1}\frac{1}{2}\sqrt{2}$; $\sin^{-1}\frac{1}{2}\sqrt{3}$; $\cos^{-1}\frac{1}{2}\sqrt{3}$; $\tan^{-1}\frac{1}{3}\sqrt{3}$.

Find the value of the following:

2. $\sin(\cos^{-1}\frac{1}{2})$.
3. $\cos(\sin^{-1}\frac{1}{2})$.
4. $\sin(\tan^{-1}1)$.
5. $\tan(\sin^{-1}\frac{1}{2}\sqrt{3})$.
6. $\sin(\sin^{-1}\frac{1}{2}\sqrt{3})$.
7. $\cos(\cos^{-1}\frac{1}{2})$.
8. $\sin(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2})$.
9. $\tan\left(\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right)$.

THEORY OF LOGARITHMS

556. **Logarithms.**—The logarithm of a number is the exponent which indicates the power to which a given base must be raised to produce the number. Thus if 100 is the number and 10 is the base, 2 is the logarithm of 100 to the base 10, for $10^2 = 100$. This relation may be written $\log_{10} 100 = 2$. If 64 is the number and 4 is the base, 3 is the logarithm of 64 to the base 4, for $4^3 = 64$. In general, if $b^x = N$, where **N** is the number, **b** the base, and **x** the logarithm, we have $\log_b N = x$.

Since $a^0 = 1$, $\log_a 1 = 0$. Hence the log of 1 to any base is 0.

Since $a^1 = a$, $\log_a a = 1$. Hence the log of any number to the same number as a base is 1.

$\mathbf{\log_b N = x}$ and $\mathbf{b^x = N}$ are inverse expressions and mean the same thing. $\mathbf{b^x = N}$ is called an exponential equation.

Note.—*An exponential equation is an equation in which the unknown occurs in the exponent.*

557. **Computation by the Use of Logarithms.**—Since a logarithm is an exponent, the computation by logarithms must be governed by the laws of exponents.

558. **To Find the Logarithm of a Product.**

The logarithm of a product is equal to the sum of the logarithms of the factors of the product.

Proof.

Let N_1 and N_2 be two numbers,

and let $\log_b N_1 = x$, (1) and $\log_b N_2 = y$. (2)

Then $N_1 = b^x$ (3) and $N_2 = b^y$. (4) §556.

Multiplying (3) by (4), $N_1N_2 = b^{x+y}$. (5)

$\therefore \log_b(N_1N_2) = x + y$. (6) §556.

Substituting the values of x and y from (1) and (2) in (6) we have,

$$\mathbf{\log_b(N_1N_2) = \log_b N_1 + \log_b N_2}. \qquad (7)$$

559. To Find the Logarithm of a Quotient.

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.

To find the logarithm of $N_1 \div N_2$.

Proof.

Dividing (3) by (4) in §558 we have,

$$N_1 \div N_2 = b^{x-y}. \qquad \text{Why?}$$

$$\therefore \log_b(N_1 \div N_2) = x - y. \qquad \text{Why?}$$

Substituting as above,

$$\mathbf{\log_b(N_1 \div N_2) = \log_b N_1 - \log_b N_2.}$$

560. To Find the Logarithm of the Power of a Number.

The logarithm of a power of a number is equal to the exponent times the logarithm of the number.

Find $\log N^m$.

Proof.

Let $\log_b N = x$. (1). Then $N = b^x$. (2)

Raising both sides of (2) to the **m** power we have,

$N^m = b^{mx}$ or $\log_b N^m = mx$. (3)

$\therefore \log_b \mathrm{N^m} = \mathrm{m} \log_b N$. (4)

561. To Find the Logarithm of the Root of a Number.

The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.

Find $\log \sqrt[m]{N}$.

Proof.

Let $\log_b N = x$. (1). Then $N = b^x$. (2)

Raising both sides of (2) to the $\frac{1}{m}$ power we have,

$$N^{\frac{1}{m}} = b^{\frac{1}{m}x} \text{ or } \log_b N^{\frac{1}{m}} = \frac{1}{m}x. \qquad (3)$$

$$\therefore \log_b N^{\frac{1}{m}} = \frac{1}{m}\log_b N. \qquad (4)$$

562. Cologarithms.—A cologarithm of a given number is a logarithm of the reciprocal of the number. That is, it is the logarithm of 1 *minus* the logarithm of the number. Thus, **colog b** = $\log \frac{1}{b}$ or **log 1 − log b.**

563. Base of a System of Logarithms.—A system of logarithms may be built up with any number as a base. There are, however, two numbers commonly used as bases of systems of logarithms. They are 10 and e. ($e = 2.718 \ldots$ §381, Prob. 10.) The base for the common or Briggian system is 10 and the base for the natural or Napierian system is e. The logarithms found in Chapter XVIII are computed to the base 10.

564. The Relation between the Common and the Natural System of Logarithms.

Let $\log_e N = x$, (1) and $\log_{10} N = y$. (2)

Then $N = e^x$, (3) and $N = 10^y$. (4)

Therefore $10^y = e^x$. (5) *Ax.* 6.

Taking the logs of both sides of (5) to the base 10 we have,

$$y = x \log_{10}e. \tag{6}$$

Substituting from (1) and (2) in (6) we have,

$$\log_{10}N = \log_e N \log_{10}e.$$

$$\therefore \log_e N = \frac{\log_{10}N}{\log_{10}e}. \tag{7}$$

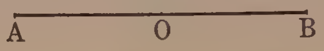
Note.—*It will be found from the tables that* $\log_{10}e = .4343$. *This is called the* **modulus** *of the common system.* $\frac{1}{.4343} = 2.3026$. *Hence, the above formula may be written.* **Natural log = 2.3026 times common log.**

SYMMETRY

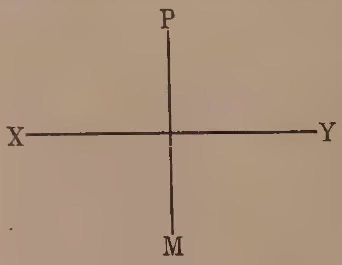
565. **Definition.**—The word symmetry is derived from two Greek words meaning **in measure with** or **having the same measure.** A figure is symmetrical when its several parts harmonize one with another, or are in due proportion and adapted to each other, as in the case of an elevation of a building in which the right and left portions, divided by an imaginary axis, are seen to be the counterparts of each other.

The idea of symmetry enters into many phases of structural engineering. Designers, artists, and architects are careful to observe the laws of symmetry in designing; in arranging the position of objects in a scene to be portrayed; and in drawing plans for a building.

566. **Symmetry of Two Points with Respect to a Third Point.**—Two points are symmetrical with respect to a third point as a center of symmetry, if the third point bisects the line joining the two points.

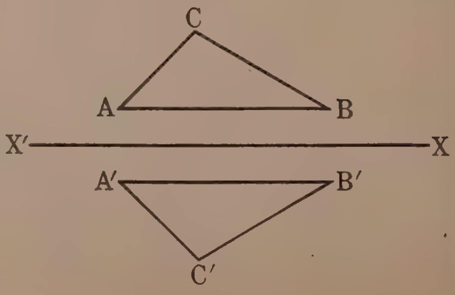


Thus, A and B are symmetrical with respect to O, for OB may be made to coincide with OA by swinging OB about on O as a center.



567. **Symmetry of Two Points with Respect to a Line.**—Two points are symmetrical with respect to a line as an axis if this axis is a perpendicular bisector of the line joining the two points.

Thus, P and M are symmetrical with respect to XY as an axis if XY is a perpendicular bisector of PM.



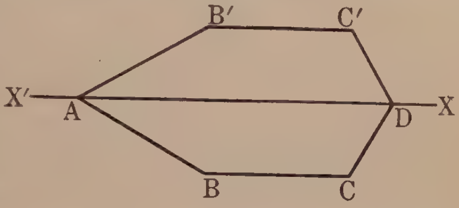
568. **Symmetry of Two Figures with Respect to a Line.**—Two figures are symmetrical with respect to a line as an axis

if every point in one figure has a correspondingly symmetrical point in the other.

Thus, the triangles ABC and $A'B'C'$ are symmetrical with respect to XX' as an axis since each point on the triangle ABC has a corresponding point of symmetry on the triangle $A'B'C'$.

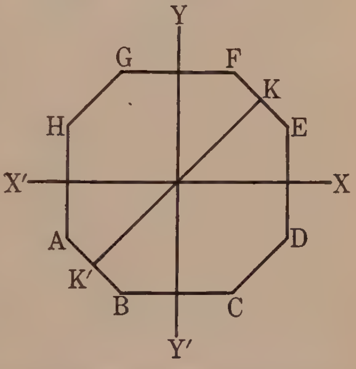
569. **Symmetry of a Figure with Respect to an Axis.**—A figure is symmetrical with respect to an axis if the axis divides the figure into two symmetrical figures.

Thus, the figure $ABCDC'B'$ is symmetrical with respect to XX' as an axis since the two figures $ABCD$ and $AB'C'D$ are symmetrical with respect to XX'.

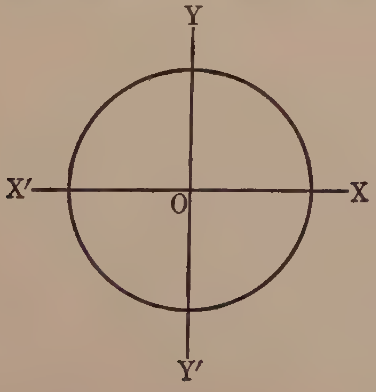


570. **Symmetry of a Figure with Respect to Two Perpendicular Axes.** If a figure is symmetrical with respect to two perpendicular axes it is symmetrical with respect to their intersection.

Thus, the figure $ABCDEFGH$ is symmetrical with respect to both axes, XX' and YY', for if the upper half of the figure is folded on XX' as an axis, it will coincide with the part below. In the same way, if YY' is used as an axis the right hand portion will coincide with the left hand portion. Again any point K on the perimeter will have a corresponding point K' such that K and K' are symmetrical with respect to O as a center.

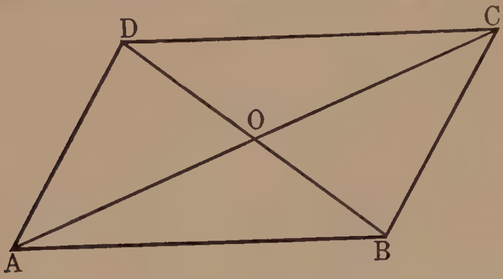


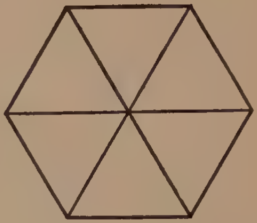
571. **Further Applications of Symmetry.**—The principle of symmetry is seen to apply to many geometric figures. A few illustrations follow.



Circle.—Here the diameters XX' and YY' are axes of symmetry. Hence their intersection O is the center of symmetry.

Parallelogram.—The parallelogram $ABCD$ is symmetrical with respect to O, the intersection of the diagonals.

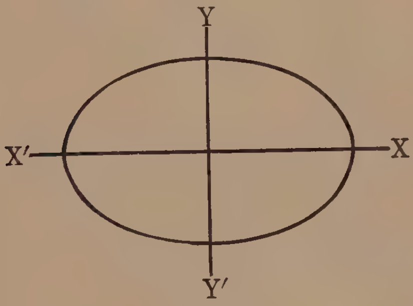




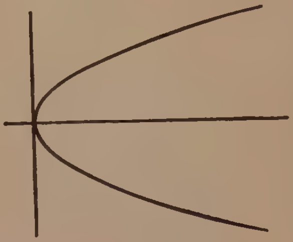
Regular Polygon.—Every regular polygon of an **even number** of sides is symmetrical with respect to its center as a center of symmetry.

PROBLEMS

1. Is the ellipse shown on the margin symmetrical with respect to the XX' and YY' as axes and to their intersection O?



PROB. 1.



PROB. 2.

2. Is the parabola symmetrical with respect to a point as a center or with respect to an axis?
3. Why can a trapezium *not* have a center of symmetry?
4. Does an isosceles trapezoid have an axis of symmetry?

SUPPLEMENTARY PROBLEMS

1. Using the formula $d = rt$, find d when:
(a) $r = 25$ and $t = 5$. (b) $r = 17.5$ and $t = 3.25$.
(c) $r = 23\frac{3}{4}$ and $t = 6\frac{1}{2}$. (d) $r = 13\frac{2}{3}$ and $t = 4\frac{5}{6}$.
2. Using the formula $d = rt$, find r when:
(a) $d = 235$ and $t = 5$. (b) $d = 137.5$ and $t = 5.5$.
(c) $d = 120$ and $t = 5\frac{2}{3}$. (d) $d = 160$ and $t = 13\frac{1}{3}$.
3. Using the formula $d = rt$, find t when:
(a) $d = 104.5$ and $r = 19$. (b) $d = 150\frac{5}{12}$ and $r = 23\frac{3}{4}$.
(c) $d = 167\frac{3}{4}$ and $r = 30\frac{1}{2}$. (d) $d = 515\frac{1}{5}$ and $r = 53\frac{2}{3}$.
4. Find the value of the following.
(a) $5xy - 3yz + 5z^2$, if $x = 3$, $y = 2$, and $z = 4$.
(b) $\frac{3mn}{q} \times \frac{mq}{n} + \frac{3n}{mq}$, if $m = 4$, $n = 3\frac{1}{2}$, and $q = 7\frac{3}{4}$.
(c) $\frac{x^3 - 3x^2 + 5x - 8y + 3}{y^3 - x^2y + 3x^3}$, if $x = 2$, and $y = 0$.

5. A ball is thrown vertically upward at the rate of 60 feet a second. The height at any time is given by the formula $s = 60t - 16t^2$. (s = distance and t = number of seconds.)

Find its height at the end of 1 sec.; $1\frac{1}{2}$ sec.; 2 sec.; $2\frac{1}{2}$ sec. Plot these relations on graph paper.

6. Which number is larger and how much?

(a) -4 or 3? (b) -17 or 18? (c) -3 or -2? (d) $-\frac{1}{10}$ or -2?
(e) $\frac{1}{2}$ or $\frac{1}{3}$? (f) $-\frac{1}{2}$ or $-\frac{1}{3}$? (g) -1 or $-1\frac{1}{2}$? (h) -3 or -3.1?

7. Find the sum of the following and check as in §31.

$3x^2 - 2y + 4$, $x^2 + 3y - 7$, $-y - 2x^2 + 6$, $5x^2 + 7y + 2$, and $-2y + 5x^2 - 8$.

8. Find the sum of the following, and check.

$ax^2 + 3y + bz^3$, $bx^2 - ay + cz^3$, $px^2 + qy - rz^3$, $rx^2 - sy - 3z^3$, and $2x + 5y - 7z$.

9. From the sum of $a + 2b - 3c$ and $3a - b + 7c$ take the sum of $-a + 4b + 7c$ and $5a - 2b + c$.

10. Subtract the sum of $x + 3y - 7z$ and $-3x - y + 8z$ from $x + y + z$.

11. Simplify, $4x - [2x - (3x - 2)]$.

12. Simplify, $7x - \{5x + [12x - (6x + 2)]\}$.

13. Simplify, $(3x - y) - [x - (3x + y) - 4x]$.

14. Multiply and check as in §31.

$$3x^2 - 5x - 4 \text{ by } 2x^2 + 3x - 7.$$

15. Multiply and check:

$$3x^3y^3 - 2x^2y^2 + xy - 4 \text{ by } 2x^2y^2 - 3xy + 5.$$

16. Divide, and check as in §31.

$$x^3 + 3x^2y + 3xy^2 + y^3 \text{ by } x + y.$$

17. Divide $x^{3m} + y^{3m}$ by $x^m - y^m$.

18. Divide, $a^5 + 31a - 7a^3 + 4a^4 - 42 - 15a^2$ by $a + 3$. Check using the formula

$$\text{Divisor} \times \text{quotient} = \text{dividend}.$$

19. Which of the following are correct?

(a) $2x^3 = 16$ if $x = 2$.
(b) $3x^2 = 24$ if $x = 4$.
(c) $3x^0 = \frac{1}{3}$ if $x = 1$.
(d) $5x^{-1} = 2\frac{1}{2}$ if $x = 2$.

20. Multiply $7ab^2c^0$ by -6^0.

21. Multiply $3a^2b^0c - 5a^0b^2c$ by $2abc^0 - 3a^0b$.

22. Divide $42a^3m^0n^2$ by $6a^2n^3$, and check.

23. Divide $12x^2yz^0$ by $-2x^3y^2z$, and check.

24. Divide 28 into two parts such that the first shall be 2 less than 5 times the second.

25. Find four consecutive numbers whose sum is 66.

26. Find four consecutive odd numbers whose sum is 112.

27. Find three consecutive even numbers whose sum is 168.

28. The height of a railroad bridge is 275 feet, which is 75 feet less than twice the width of the span. What is the width of the span?

29. The hoist shaft of a mine is 700 feet deep. If the shaft is 25 feet more than 15 times the height of the boiler stack, how high is the boiler stack?

30. The central arch of a 7 arch bridge is 95 feet longer than each of the six other arches. The total length of the bridge is 585 feet. Find the length of each arch.

31. Divide $4200 among A, B, and C so that B receives $800 less than A and C receives $1000 less than A and B together.

32. How many degrees in each of two supplementary angles if one angle is 22° 30′ larger than the other?

33. Each angle of an equiangular polygon is $154\frac{2}{7}°$. How many sides has the polygon?

34. Can a triangle be constructed using the sides 10.2″, 11.7″ and 22″?

35. What is the sum of the interior angles of a polygon of 11 sides? Of the exterior angles?

36. Find the length of the legs of an isosceles triangle if the perimeter is 217″ and the base is 53″.

37. Divide a given line into 5 equal parts.

38. Locate a point 15 feet from a given line and 35 feet from a given point. How many solutions?

39. The acute angles of a right triangle are 30° and 60°. The hypotenuse is 21.72″. Find the shorter leg.

40. $ABCD$ is a parallelogram. E is the mid-point of the side AB. The diagonal AC and the line DE intersect in a point 0. Prove $AO = \frac{1}{3}AC$.

Find the prime factors of the following:

41. $4y^3 - 2y$.
42. $3x^2 - 9x^3y$.
43. $ar - br + as - bs$.
44. $4m^2x^2 - 20mx - 252$.
45. $3z^2 - 3x^2 - 6xy - 3y^2$.
46. $a^3 + b^3$.
47. $1 + 8y^3$.
48. $7x^2 + 48x - 7$.
49. $m^2 + 64$.
50. $2r^4 + 128$.
51. $r^3 - (m + n)^3$.
52. $r^3 + (m - n)^3$.
53. $r^3 - 8(r + s)^3$.
54. $4x^2 - y^2 - 9 - 6y$.
55. $m^4 + 6m^2 + 25$.
56. $6pq + 14pc - 21q - 49c$.
57. $\frac{1}{2}y^3 - \frac{7}{2}y^2 + 6y$.
58. $.25x^5 - 16x^3$.
59. $x^{4n} - 2x^{2n} - 3$.
60. $y^{5n} + 2y^{3n} - 3y^n$.

Factor, using the Remainder Theorem.

61. $x^3 - 2x^2 - 5x + 6$.
62. $x^3 - 3x^2 - 6x + 8$.
63. $y^3 + 2y^2 - 5y - 6$.
64. $m^3 - m^2 - m + 1$.
65. $r^3 - 6r^2 + 11r - 6$.
66. $2p^3 - p^2 - 5p - 2$.
67. $q^3 - 3q + 2$.
68. $a^3 + a^2 - a - 1$.
69. $b^3 + 5b^2 - b - 5$.
70. $a^4 + 5a^3 + 5a^2 - 5a - 6$.

71. Tell which of the following are correct. Give reasons.

(a) $-\frac{4}{9} = \frac{-4}{9}$.
(b) $\frac{x-y}{y} = \frac{y-x}{y}$.
(c) $\frac{x-y}{y} = \frac{y-x}{-y}$.

(d) $\frac{3-r}{2} = \frac{r-3}{2}$.
(e) $\frac{m-n}{-a} = \frac{n-m}{-a}$.
(f) $\frac{-(b-c)}{b} = -\frac{c-b}{b}$.

(g) $\frac{(a-b)(b-c)}{c-d} = \frac{(b-a)(c-b)}{d-c}$.
(h) $\frac{(m-n)(n-q)}{r-s} = \frac{(n-m)(n-q)}{r-s}$.

(i) $\frac{(x+y)(y-z)}{z-x} = \frac{(y+x)(z-y)}{x-z}$.
(j) $\frac{(x-y)(y-z)}{z-x} = \frac{(z-y)(y-x)}{z-x}$.

Simplify the following.

72. $\frac{y}{2+y} - \frac{y^2}{y^2-4} - \frac{y}{2-y}$.
73. $\frac{a}{a+b} + \frac{b}{a-b} + \frac{a^2+b^2}{b^2-a^2}$.
74. $\frac{x}{x+2} - \frac{2}{x-2} - \frac{4}{4-x^2}$.
75. $\frac{a-5}{a^2+5a+6} - \frac{15a}{9-a^2}$.
76. $\frac{2a+b}{2y+2a} - \frac{2b-a}{a-y} - \frac{1}{y^2-a^2}$.
77. $\frac{x+1}{6x-6} + \frac{2}{3-3x^2}$.
78. $\frac{a^2-12a+32}{21-10a-a^2} \times \frac{a^2-11a+24}{a^2-a-56} \div \frac{a^2-7a+12}{a^2-49}$.
79. $\dfrac{x+4-\frac{5}{x}}{x-2-\frac{35}{x}}$.
80. $\dfrac{m+n-\frac{n^2}{n-m}}{m-n-\frac{m^2}{m+n}}$.

81. What is the value of the fraction $\frac{1}{x}$, when $x = 1$? When $x = 10$? When $x = 100$? When $x = 1000$? What effect does an increase in the denominator of a fraction have upon the value of a fraction, if the numerator remains constant?

82. What is the value of the fraction $\frac{1}{x}$, when $x = 1$? When $x = .1$? When $x = .01$? When $x = .001$? What effect does a decrease in the denominator of a fraction have upon the value of the fraction, if the numerator remains constant?

83. Given the fraction $\frac{x}{y}$. What change occurs in the value of the fraction if x increases and y remains constant? What change occurs, if x decreases and y remains constant?

84. Which of the following are correct?

(*a*) $\frac{1}{a} + \frac{1}{b} = \frac{ab}{a+b}$. (*b*) $\frac{1}{x} + \frac{1}{y} = \frac{2}{xy}$.

(*c*) $\frac{1}{m} - \frac{1}{n} = \frac{m-n}{mn}$. (*d*) $\frac{x-y}{y-x} + \frac{y+x}{x+y} = 0$.

85. If $A = \pi r^2$, does $\pi = Ar^2$?

86. If $A = \frac{1}{2}bh$, does $h = \frac{b}{2A}$?

87. If $A = \pi r^2$, does $r^2 = \frac{A}{\pi}$?

88. If $A = p + prt$, does $p = \frac{A}{1 + rt}$?

89. If $C = \frac{5}{9}(F - 32°)$, does $F = \frac{5}{9}(C + 32°)$?

Solve for x, and check.

90. $\frac{4x}{9} - \frac{5x}{6} = \frac{4}{3} - \frac{3x}{2}$.

91. $\frac{3x-1}{4} - \frac{2x+1}{3} - \frac{4x-5}{5} = 4$.

92. $5 - \left(\frac{3x-1}{4} + \frac{2x+1}{3}\right) = 4 - \left(\frac{2x-5}{3} + \frac{7x-1}{8}\right)$.

93. $\frac{12x^2 - 2x + 5}{4x^2 - 3x + 2} = 3$. 94. $\frac{2x-1}{2} - \frac{x+2}{2x+5} = \frac{6x-5}{6}$.

95. $\frac{2x+1}{2x-1} - \frac{2x-1}{2x+1} = \frac{8}{4x^2-1}$.

96. $\frac{x+2}{x-2} - \frac{2x+5}{x+3} = \frac{8-x^2}{x^2+x-6}$.

97. $\frac{a+x-3}{a-x+3} = \frac{b}{c}$. 98. $\frac{5ax}{a-b} - 8x = 3a$.

99. Given $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$, solve for R, r_1, and r_2.

100. Solve for W in the formula $x = \frac{4WV^2}{5R}$.

(*a*) When $x = 1.5$, $V = 40$, and $R = 2000$.
(*b*) When $x = 2.72$, $V = 70$, and $R = 5000$.

101. One formula for computing the H.P. of an automobile engine is, H.P. $= \frac{D^2N}{2.5}$ in which N is the number of cylinders, and D is the diameter of each cylinder (in inches). Find the H.P. of a six cylinder engine if the diameter of each cylinder is $3\frac{1}{4}$ inches.

102. What is the brake horsepower (B.H.P.) developed by an engine when the length of the brake arm is 60″, tare weight on brake 20 pounds, total pressure on the brake arm 156 pounds, and the brake wheel makes 200 revolutions?

B.H.P. $= \frac{2\pi LWN}{33000}$, where L = length of brake arm in feet, W = net presure *i.e.*, total pressure on brake arm less tare weight in pounds, and N = number of revolutions per minute.

103. How many gallons of water are delivered per stroke by a 3 plunger pump, each 8″ in diameter with a 12″ stroke, neglecting slip?

$G = \frac{nAl}{231}$, where n = number of plungers, A = cross-section of each plunger in sq. inches, and l = length of stroke in inches.

104. Divide 54 into two parts such that $\frac{2}{3}$ of the greater minus $\frac{3}{8}$ of the smaller shall equal 11.

105. The supplement of a certain angle divided by 3 times its complement equals $\frac{5}{3}$. Find the angle.

106. The difference between two numbers is 26. If the greater is divided by the smaller, the quotient is 3, and the remainder is 2. Find the numbers.

107. A belt connects two wheels whose circumferences are 4 and $7\frac{1}{2}$ feet, respectively. How many feet will a point on the belt have traveled when the smaller wheel has made 100 more revolutions than the larger wheel?

108. The sum of two numbers is m and the difference is n. Find the numbers.

109. A can do a piece of work in x days and B, in y days. In how many days can they do it, working together?

110. A can do a piece of work in 3 days and B, in 5 days. In how many days can the work be completed, if they work together? Check your result by substituting in the answer of the preceding problem.

Solve for x and check.

111. $\frac{.2x}{3} - \frac{.3x}{5} = \frac{2}{5}$.

112. $\frac{.15x}{4} - \frac{.25x}{3} = .55$.

113. $\frac{3x}{.2} - \frac{5x}{.6} = \frac{4}{.3}$.

114. $\frac{2.5x + .3}{.2} - \frac{5.2x - .5}{.5} = 13$.

115. If $a : b = c : d$, show that $a + 5b : a - 5b = c + 5d : c - 5d$.

116. Find x if $6x - 5a : 4x - 5b = 3x + 5b : 2x + 5a$.

117. If w varies inversely as P, and w is 8 when P is 12, find w when P is 24.

118. Two chords intersect within a circle and the arcs intercepted between the sides of a pair of vertical angles are 45° and 75° respectively. Find the angles formed by the chords.

119. The triangle ABC is inscribed in a circle. D is the midpoint of the arc AC, and BD intersects the line AC in K. Prove that $AB : BC = AK : KC$.

120. AB is the diameter of a circle O, BP is a tangent, and AP intersects the circumference at Q. Prove $\Delta\ ABQ$ similar to $\Delta\ APB$ and $AQ : AB = BQ : BP$.

121. AB is a chord in a circle, CD the diameter perpendicular to AB and meeting it at E. CP is any chord intersecting AB in F. Prove that $CD : CF = CP : CE$.

122. Given the triangle ABC, $AD \perp BC$, and $CF \perp AB$. Prove that $AB : BC = AD : CF$.

123. Corresponding medians in two similar triangles have the same ratio as any two corresponding sides.

$$\frac{2x}{3} - \frac{3y}{4} = \frac{1}{2},$$

124. Solve for x and y and check.

$$\frac{5x}{6} + \frac{7y}{2} = \frac{19}{2}.$$

125. $\frac{3}{2x} + \frac{4}{3y} = 7,$

$$\frac{5}{2x} - \frac{7}{3y} = -2.$$

126. $\frac{x}{a} + \frac{y}{b} = 6,$

$$\frac{2x}{3a} - \frac{3y}{2b} = 3.$$

$$2x + 3y - 4z = 10,$$

127. Solve for x, y, and z and check.

$$4x - 2y + 3z = 21,$$

$$-3x + 4y + 5z = 16.$$

128. $\frac{.5x}{3} + \frac{.6y}{4} = \frac{19}{30},$

$$\frac{.4x}{3} - \frac{.2y}{5} = \frac{14}{75}.$$

129. If the greater of two numbers is divided by the smaller, the quotient is 2 and the remainder 8; but if nine times the smaller is divided by twice the greater, the quotient is 1 and the remainder, 54. Find the numbers.

130. A certain fraction becomes $1\frac{1}{5}$ when the numerator is increased by 4, and $\frac{2}{3}$ when the denominator is decreased by 2. Find the fraction.

131. The sum of the two digits of a number is 9. If the number is divided by the sum of the digits, the quotient is 4. Find the number.

132. A number is expressed by three digits. If the digits are reversed, the new number exceeds the original number by 198. If the number is divided by the sum of its digit, the quotient is 26 and the remainder, 1. The sum of the tens' and units' digits exceeds the hundreds' digit by 8. Find the number.

133. A man rows 24 miles down a river and back in 10 hours. He rows 6 miles down the river in the same time that he rows 4 miles up the river. Find his rate in still water, the rate of the current, the time he was rowing down, and the time he was rowing up the river.

134. If the length of a rectangular field is increased 4 rods and its width, 3 rods, its area is increased 120 square rods. If the length is decreased 4 rods and the width, 2 rods, the area is diminished 80 square rods. Find the length and width of the field.

Which of the statements in problems 135 to 144 inc. are correct?

135. $(x + \sqrt{3})^2 = x^2 + 3$.
136. $(2\sqrt{x} + 1)^2 = 2(x + 1)$.
137. $3\sqrt{x^2 - 5} = 9x^2 - 45$.
138. $(2\sqrt{x} - 1)^2 = 4x - 1$.
139. $(x + y)^2 = x^2 + y^2$.
140. $\sqrt{x} - \sqrt{y} = \sqrt{x - y}$.
141. $\sqrt{xyz} = \sqrt{x}\cdot\sqrt{y}\cdot\sqrt{z}$.
142. $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$.
143. $\sqrt{x^2 - y^2} = x - y$.
144. $2a^{-2} - x^{-3} = \frac{1}{2a^2} - \frac{1}{x^3}$.

145. Multiply $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

146. Divide $a + b$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

147. Express with rational denominators.

(a) $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$ (b) $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.

148. Simplify: $\sqrt{32a} - 4\sqrt{\frac{a}{2}} + 2\sqrt{\frac{9a}{2}} - 3\sqrt{8a}$.

149. Solve for x and check. $\sqrt{x + 5} - \sqrt{x} = 1$.

150. Expand $\left(3x - \frac{y}{3}\right)^4$.

151. Expand $(\sqrt{3} - \sqrt{2})^3$.

152. Find the square root of 2867.52 correct to two decimal places.

153. Find the square root of $9x^4 + 34x^2 - 12x^3 + 25 - 20x$.

154. Find the cube root of 15813251.

155. Find the fourth root of $16x^4 - 32x^3 + 24x^2 - 8x + 1$.

156. Solve and check.

$$\frac{x^2 + 9}{3} - \frac{2x^2 - 5}{6} + \frac{x^2 + 10}{2} = \frac{40}{3}.$$

157. Solve and check.

$$\frac{3x + 5}{x - 2} - \frac{2x - 5}{x + 2} = \frac{19}{3}.$$

158. Using the quadratic formula, solve $mx^2 - nx + p$ for x.

159. Solve and check. $x - 5x^{\frac{1}{2}} + 6 = 0$.

160. Solve and check. $2x - \sqrt{3x - 2} = 8$.

161. Solve and check. $\sqrt{x + 2} - \sqrt{3x + 4} + 2 = 0$.

162. Solve and check. $\sqrt{2x + 5} - \sqrt{3x - 2} = \sqrt{5x - 9}$.

163. Solve and check. $2x + \sqrt{2x + 3} = 9$.

164. Solve $(x^2 + 3x)^2 - 2x^2 - 6x = 15$.

165. Form the equation whose roots are $a + 2\sqrt{b}$ and $a - 2\sqrt{b}$.

166. Find two consecutive odd integers whose product is 143.

167. The sum of two numbers is 11 and their product is 30. Find the numbers.

168. The quotient of two numbers is $1\frac{1}{2}$ and their product is 216. Find the numbers.

169. The difference between the hypotenuse and one side of a right triangle is 4, and the difference between the hypotenuse and the other side is 8. Find the sides.

170. Solve for x and y.

$$3x^2 - 2y^2 = -6,$$
$$5xy + 3y^2 = 57.$$

171. The cylinder of a pump is 12″ long and 9″ in diameter. What is the diameter of a cylinder 15″ long having the same volume?

172. The volume of air flowing through a mine varies directly as the square root of the pressure. If the flow of air is 60,000 cu. ft. a minute under a pressure of 2.8 lbs. a sq. ft. what pressure is required to supply a flow of 70,000 cu. ft. a minute.

173. If 18 men can build 2079 feet of R.R. siding in 60 days, working 10 hrs. a day, how many feet can 6 men build in 110 days, working 12 hrs. a day?

174. The volume of a cylinder varies directly as the product of the length by the square of the diameter.

If the volume of a gas engine cylinder 20″ × 10″ is 1570.8 cu. in., what is the volume of a gas engine cylinder 24″ × 16″?

Find the value of the unknown angle in the following trigonometric equations. Check.

175. $(2 \cos x - 1)(\cos x - 1) = 0$.

176. $3 \cos \theta + 2 \sin^2 \theta = 3$.

177. $\tan^2 y = 3 \sec y - 3$.

178. $2 \cos x \sin^2 x - \cos x = 0$.

179. $\sin 2B = \cos B$.

180. $\csc 5x = \sec 3x$.

181. $\sin (60° + \theta) = \cos 2\theta$.

182. $6 \sin \theta + 4 \cos 2\theta = 5$.

183. $2 \sin \theta + 2 \sin 2\theta - 2 \cos \theta = 1$.

184. $2 \sin \theta + \tan 2\theta = 0$.

185. One leg of a right triangle is 4.8′, the hypotenuse is 7.3′. (*a*) Find the other leg. (*b*) Find the six functions of each acute angle.

186. To give the greatest support to a pole, a guy wire of given length must be fastened at an angle of 45° with the pole. Assuming the ground to be level, how high up the pole must a 24′ guy wire be placed to give the greatest strength? (*b*) What is the shortest distance from the foot of the pole to the guy wire?

187. A road bed is inclined +150° to the horizontal. If a person walks up the road 1 mile, how many feet has he risen, vertically?

188. In problem 187 how many feet has he gone horizontally?

189. If the person weighs 200 lbs., how many foot-pounds of work has he done in problem 187?

190. In a circle with a radius of 12.975 inches, find the central angle subtended by a chord 4.063 inches long.

Solve the following oblique triangles having given:

191. $b = 41.554$, $A = 118° 55' 49''$, and $C = 45° 41' 35''$.

192. $a = .007632$, $b = .004793$, and $C = 102° 3' 42''$.

193. $a = .34678$, $b = .91704$, $c = .80006$.

194. $a = 365.24$, $c = 346.45$, $C = 62° 17' 48''$.

195. Two trains start at the same time from the same point and upon straight tracks diverging at an angle of 67° 45′. If the one runs 46 miles an hour and other 32 miles an hour, how far apart are they at the end of three hrs.?

196. In any circle, the angle at the center subtended by a chord 100 feet long is called the degree of curve. If the central angle is denoted by D and the radius by R, show that $\sin \frac{D}{2} = \frac{50}{R}$.

197. If the radius of a curve is 500 feet, find the degree of curve.

198. Find the radius of a curve (in feet) having a 25 degree curvature.

199. If the radius of a curve is 1000 feet, find the degree of curve.

200. Find the radius of a curve (in feet) having a 15 degree curvature.

SUMMARY OF FORMULAS

Notation

a, b, and c = the sides of a triangle.
$s = \frac{1}{2}(a + b + c)$.
h = altitude.
h_c = altitude on side c.
m = median.
m_c = median to side c.
t = angle bisector.
t_c = angle bisector meeting side c.
A = area.
b = base.
d = diagonal.
p = perimeter.
C = circumference of a circle.
R = radius of a circle or a sphere.
D = diameter.
r = radius of an inscribed circle.
V = volume.
l, w = length, width.
B = area of base.
M = area of mid-section.
$L.A.$ = lateral area.
$T.A.$ = total area.
s = slant height or element.
N = number.

Binomial Theorem.

1. $(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \dots$ §381.

Quadratic.

If $ax^2 + bx + c = 0$,

2. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. §394.

Triangles.

3. $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. §433.
4. $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$. §434.
5. $t_c = \frac{2}{a+b}\sqrt{abs(s-c)}$. §435.
6. $A = \frac{1}{2}bh$. §430.
7. $A = \sqrt{s(s-a)(s-b)(s-c)}$. §444.
8. $A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$. §549–1.

9. $A = \frac{a^2 \sin B \sin C}{2 \sin A}.$ §549–2.

10. $A = \frac{a^2\sqrt{3}}{4}.$ §443.

Quadrilaterals.

11. $A = s^2.$ Square. §427.

12. $A = bh.$ Parallelogram. §429.

13. $A = \frac{1}{2}dd_1.$ Rhombus. §431.

14. $A = \frac{1}{2}h(b + b_1).$ Trapezoid. §432.

Regular Polygons.

15. $A = \frac{1}{2}hp.$ §467.

Circles.

16. $C = 2\pi R.$ §466.

17. $A = \frac{1}{2}RC.$ §468.

18. $A = \pi R^2.$ §470.

19. $R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$ §436.

20. $R = \frac{abc}{4A}.$ §445.

21. $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$ §446.

SOLIDS

Rectangular Solid.

22. $V = lwh.$ §482.

Prisms.

23. $V = Bh.$ §485

24. $L.A. = ph.$ §487.

25. $T.A. = L.A. + 2B.$ §488.

Pyramids.

26. $V = \frac{1}{3}Bh.$ §490.

27. $L.A. = \frac{1}{2}sp.$ §491.

28. $T.A. = L.A. + B.$ §492.

Prismatoids.

29. $V = \frac{1}{6}h(B + B_1 + 4M).$ §495.

Cylinders.

30. $V = Bh.$ §498.

31. $L.A. = Ch.$ §499.

32. $T.A. = L.A. + 2B.$ §500.

Cones.

33. $V = \frac{1}{3}Bh.$ §503.

34. $L.A. = \frac{1}{2}Cs.$ §504.

35. $T.A. = L.A. + B.$ §505.

Spheres.

36. $A = 4\pi R^2.$ §509.

37. $A = 2\pi Rh.$ §510.

38. $V = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3.$ §511.

39. $V = \frac{h}{2}(\pi r_1^2 + \pi r_2^2) + \frac{\pi h^3}{6}.$ §512.

40. $V = \frac{1}{2}\pi r^2 h + \frac{1}{6}\pi h^3.$ §512.

TRIGONOMETRIC FORMULAS

41. $\sin^2 A + \cos^2 A = 1.$ §518

42. $\tan A = \frac{\sin A}{\cos A}.$

43. $\sec^2 A = 1 + \tan^2 A.$

44. $\csc^2 A = 1 + \cot^2 A.$

45. $\sin(90^\circ + x) = \cos x.$ §533–I.

46. $\cos(90^\circ + x) = -\sin x.$

47. $\tan(90^\circ + x) = -\cot x.$

48. $\sin(180 - x) = \sin x.$ §533–II.

49. $\cos(180^\circ - x) = -\cos x.$

50. $\tan(180^\circ - x) = -\tan x.$

51. $\sin(x + y) = \sin x \cos y + \cos x \sin y.$ §534.

52. $\cos(x + y) = \cos x \cos y - \sin x \sin y.$

53. $\tan(x + y) = \frac{\tan x + \tan y.}{1 - \tan x \tan y}.$

54. $\cot(x + y) = \frac{\cot y \cot x - 1}{\cot y + \cot x}.$

55. $\sin(x - y) = \sin x \cos y - \cos x \sin y.$ §535.

56. $\cos(x - y) = \cos x \cos y + \sin x \sin y.$

57. $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$

58. $\cot(x - y) = \frac{\cot y \cot x + 1}{\cot y - \cot x}.$

59. $\sin 2x = 2 \sin x \cos x.$ §536.

60. $\cos 2x = \cos^2 x - \sin^2 x.$

61. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$

62. $\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$

63. $\sin \frac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{2}}.$ §537.

64. $\cos \theta\frac{1}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}.$

65. $\tan \frac{1}{2}\theta = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$

66. $\cot \frac{1}{2}\theta = \sqrt{\frac{1 + \cos\theta}{1 - \cos\theta}}.$

67. $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$ §538.

68. $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$

69. $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$

70. $\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$

71. $\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$

72. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$ §540.

73. $\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{a - b}{a + b}.$ §541.

74. $a^2 = b^2 + c^2 - 2bc \cos A.$ §542.

75. $(a - b) \cos \frac{1}{2}C = c \sin \frac{1}{2}(A - B).$ §543.

76. $\tan \frac{1}{2}A = \frac{r}{s - a}, \tan \frac{1}{2}B = \frac{r}{s - b}, \tan \frac{1}{2}C = \frac{r}{s - c}.$ §548.

CIRCULAR MEASURE

77. No. of radians $\times \frac{180}{\pi}$ = No. of degrees. §552.

78. No. of degrees $\times \frac{\pi}{180}$ = No. of radians.

LOGARITHMS

79. $\log_b (N_1 N_2) = \log_b N_1 + \log_b N_2.$ §558.

80. $\log_b (N_1 \div N_2) = \log_b N_1 - \log_b N_2.$ §559.

81. $\log_b N^m = m \log_b N.$ §560

82. $\log_b N^{\frac{1}{m}} = \frac{1}{m} \log_b N.$ §561.

83. $\operatorname{colog}_b N = \log_b 1 - \log_b N.$ §562.

84. $\log_e N = \frac{\log_{10} N}{\log_{10} e}.$ Where $e = 2.718.$ §564.

i.e. Natural log N = 2.3026 times common log N.

APPROXIMATE VALUES

$\sqrt{2} = 1.4142.$

$\sqrt{3} = 1.7321.$

$\sqrt{5} = 2.2361.$

$\sqrt{\tfrac{1}{2}} = 0.7071.$

$\sqrt[3]{2} = 1.2599$

$\pi = 3.1416$

$\frac{1}{\sqrt{\pi}} = 0.5642.$

$\pi^2 = 9.8696.$

$\sqrt{\pi} = 1.7725.$

$\frac{1}{\pi} = 0.3183.$

$1° = .01745^r$ and $1^r = 57° \, 17' \, 45''.$

CHAPTER XVIII

TABLES

LOGARITHMS OF NUMBERS

572. **Base.**—As has been explained in §563 the base of the logarithms computed in the tables is 10.

Now $10^0 = 1$, hence $\log 1 = 0$ §556.
$10^1 = 10$, hence $\log 10 = 1$.
$10^2 = 100$, hence $\log 100 = 2$.
$10^3 = 1000$, hence $\log 1000 = 3$
$10^4 = 10000$, hence $\log 10000 = 4$.
$10^{-1} = .1$, hence $\log .1 = -1$.
$10^{-2} = .01$, hence $\log .01 = -2$.
$10^{-3} = .001$, hence $\log .001 = -3$, etc.

It will be seen from the above that:

1. The logarithm of a number greater than 1 is positive.
2. The logarithm of a number between 0 and 1 is negative.
3. Since the logarithm of 1 is 0 and the logarithm of 10 is 1, the logarithm of a number between 1 and 10 is a positive decimal fraction.
4. Since the logarithm of 10 is 1 and the logarithm of 100 is 2, the logarithm of a number between 10 and 100 is 1 plus a decimal fraction.
5. Since the logarithm of 100 is 2 and the logarithm of 1000 is 3, the logarithm of a number between 100 and 1000 is 2 plus a decimal fraction.

It will be further seen that:

6. Since the logarithm of 1 is 0 and the logarithm of 0.1 is -1, the logarithm of a number between 1 and 0.1 is -1 plus a decimal fraction.
7. Since the logarithm of 0.1 is -1 and the logarithm of 0.01 is -2, the logarithm of a number between 0.1 and 0.01 is -2 plus a decimal fraction, etc.

Most numbers are not integral powers of 10. Thus, 780 is more than the second and less than the third power of 10. Hence its logarithm is 2 plus a fraction.

573. **Characteristic.**—The logarithm of a number consists of two parts, the **characteristic** and the **mantissa.** The characteristic is the whole number as determined in §572. It is **not** found in the tables but must be supplied by the student. The positive characteristic is determined by the number of digits in the given number and is always numerically **one** less than the number of digits to the left of the decimal point. The logarithm of every number between 0 and 1 has a negative characteristic. This characteristic is numerically **one** more in absolute value than the number of **zeros** between the decimal point and the first significant digit. (All digits except **zero** are significant digits.)

If the characteristic of a logarithm is negative, it is customary in computation to add and subtract 10. Thus, the logarithm -1 plus a decimal fraction is changed to read 9 plus a decimal fraction -10, since -1 is equal to $9 - 10$.

Illustrations.

log 32 = 1 + a decimal fraction.
log 320 = 2 + the same decimal fraction.
log 3.2 = 0 + the same decimal fraction.
log .32 = 9 + the same decimal fraction -10.
log .032 = 8 + the same decimal fraction -10.

PROBLEMS

What is the characteristic of the logarithm of each of the following numbers?

1. 485.
2. 3.72.
3. 562.
4. .385.
5. .0452.
6. 48620.
7. 679.54.
8. .00482

574. **Mantissa.**—The mantissa is the decimal fraction referred to in the preceding article. Tables give the mantissa only. It depends upon the sequence of figures in the number. It also determines the sequence of figures in the number corresponding to a given logarithm.

In the log, 3.68542, the whole number 3 is the characteristic and the decimal part, .68542, is the mantissa. The decimal point is omitted in the tables.

575. **To Find the Logarithm of a Number.**—If the number consists of not more than three digits it will be found in the column headed No., and the mantissa of its logarithm will be found horizontally opposite in the first column to the right. If, however, the number consists of four digits, the fourth digit will be found to the right of No., and the mantissa of its logarithm is found in the column under the fourth digit and to the right of the first three digits in the number.

Check the following from the tables:

$$\log 32 = 1.50515.$$
$$\log 324 = 2.51055.$$
$$\log .368 = 9.56585 - 10.$$
$$\log 5.423 = 0.73424.$$
$$\log .04867 = 8.68726 - 10.$$

576. **Antilogarithm.**—An antilogarithm is a number which corresponds to a given logarithm. If the logarithm of a number is given to find the number, look in the tables for the mantissa and write down the number found to the left.

Check the following from the tables:

If $\log x = 1.73656.$ $x = 54.52.$
If $\log x = 3.68178.$ $x = 4806.$
If $\log x = 0.90325.$ $x = 8.003.$
If $\log x = 8.84621 - 10.$ $x = .07018.$

PROBLEMS

Find the logarithm of the following numbers:

1. 862.
2. 6851.
3. 76420.
4. 54.63.
5. 3086.
6. 954.3.
7. 2.367.
8. .1468.
9. .03845.
10. .00642.
11. .00048.
12. 6.386.

Find x in the following:

13. $\log x = 0.64473$.
14. $\log x = 1.48087$.
15. $\log x = 3.39023$.
16. $\log x = 2.97109$.
17. $\log x = 9.89818 - 10$.
18. $\log x = 8.78433 - 10$.

577. Interpolation.—It is often necessary to get the logarithm of a number of more than four figures. In order to do this, a correction must be made as follows. First get the logarithm of the number of the first four figures and also that of the next higher number. Multiply the difference of these logarithms by the fifth figure used as tenths and add the result to the logarithm of the smaller number.

Illustrations.—1. To find log 74875.

$$\log 74870 = 4.87431.$$
$$\log 74880 = 4.87437.$$

Then log 74875 lies between 4.87431 and 4.87437.

The required log is found by interpolation as follows:

The tabular difference is .00006.

The difference between 74870 and 74875 is 5.

The difference between 74870 and 75880 is 10.

Therefore, $\frac{5}{10}$ of .00006 which is .00003 must be added to the log of 74870.

Then log 74875 = 4.87434.

In practice the zeros in the differences are omitted.

2. Given $\log x = 1.74458$, find x.

$$\log 55.53 = 1.74453.$$
$$\log 55.54 = 1.74461.$$

Since $\log x$ lies between log 55.53 and log 55.54, x must have a value greater than 55.53 and less than 55.54.

The difference between $\log x$ and log 55.53 is 5. The tabular difference is 8. Therefore $\frac{5}{8}$ must be added to 55.53 making the required number x 55.53 $\frac{5}{8}$ or 55.536.

Hence $x = 55.536$.

PROBLEMS

Find the logarithm of the following numbers.

1. 23854.
2. 65.402.
3. 9.0264.
4. 383.67.
5. 1104.5.
6. .46843.
7. 517.06.
8. .074321.
9. .86389.
10. .0023105.
11. .084653.
12. .00057207.

Find x in the following:

13. $\log x = 1.73275$.
14. $\log x = 3.84251$.
15. $\log x = 2.90332$.
16. $\log x = 0.99381$.
17. $\log x = 7.09969 - 10$.
18. $\log x = 9.75589 - 10$.

578. **Use of Co-logarithm.**—The co-logarithm of a number is found by subtracting its logarithm from the logarithm of 1. §562.

Thus, **a** **b**

$$\begin{array}{rll} \log 1 = & 10.00000 - 10 & \quad 9.9999(10) - 10 \\ \log 53.84 = & 1.73111 & \quad 1.73111 \\ \hline & \text{or} & \\ \text{Colog } 53.84 = & 8.26889 - 10 & \quad 8.26889 \quad - 10. \end{array}$$

If the above subtraction as shown in method **b** is used, the colog may be read directly from the table.

Illustrations 1.

Find the value of $\frac{3.467}{5.384}$ by using co-logs.

$$\begin{array}{rl} \log 3.467 = & 0.53995 \\ \textbf{colog 5.384} = & \mathbf{9.26889 - 10} \\ \text{Adding, } \log \text{quotient} = & 9.80884 - 10 \\ \text{anti} \log = & 0.64393. \end{array}$$

2. Find the value of: $\frac{312 \times 7.18 \times 31.82}{519 \times 8.27 \times 5.132}$.

$$\begin{array}{lrl} \log & 312 = & 2.49415 \\ \log & 7.18 = & 0.85612 \\ \log & 31.82 = & 1.50270 \\ \text{colog} & 519 = & 7.28483 - 10 \\ \text{colog} & 8.27 = & 9.08249 - 10 \\ \text{colog} & 5.132 = & 9.28971 - 10 \\ \hline \end{array}$$

$$\begin{array}{rl} \text{Adding, } \log Ans. = & 0.51000 \\ Ans. = & 3.2360. \end{array}$$

3. Find the value of: $\frac{\sqrt[3]{3.256} \times 685^{\cdot 2}}{\sqrt[5]{68.44} \times 42.867^{\frac{3}{5}}}$.

$$\begin{array}{lrl} \log & 3.256 = & 0.51268 \\ \log & 685 = & 2.83569 \\ \log & 68.44 = & 1.83531 \\ \log & 42.867 = & 1.63212 \\ \hline \end{array}$$

$$\begin{array}{lrl} \log & \sqrt[3]{3.256} = & 0.17089 \\ \log & 685^{\cdot 2} = & 0.56714 \\ \text{colog} & \sqrt[5]{68.44} = & 9.63294 - 10 \\ \text{colog} & 42.867^{\frac{3}{5}} = & 9.02073 - 10 \\ \hline \end{array}$$

$$\begin{array}{rl} \text{Adding, } \log Ans. = & 9.39170 - 10 \\ Ans. = & .24643. \end{array}$$

PROBLEMS

Find the value of the following:

1. $\frac{3864}{1538}$. 2. $\frac{2.687}{.9864}$. 3. $98^{\frac{2}{5}}$. 4. $\sqrt[8]{216}$. 5. $4^{\frac{5}{4}}$.

6. $\frac{(-.87324) \times 3.84}{(-.09324) \times 56.2}$. 7. $\sqrt[3]{\frac{5.623 \times .028634 \times 879.23}{2.8645 \times .9872 \times 5.8974}}$.

8. $\sqrt[9]{6384.2 \times .02674}$. 9. $\frac{29.86^{\frac{3}{5}}}{\sqrt[3]{42.86 \times 3.876}}$.

10. $2568.4^{.7}$. 11. $(-286.32)^4 \times .00976$.

12. $\frac{67254^{.2} \times 8643^{.8}}{672.4^{.5}}$. 13. $\sqrt[5]{(3752)^3} \div 2753^{.4}$.

14. $\left(\frac{.3784^{.35} \times 5.462^{.75}}{\sqrt[6]{4.3875}}\right)^{\frac{4}{3}}$.

579. **Logarithmic Functions.**—Only the logarithms of sine, cosine, tangent, and cotangent of the acute angles are given in the tables. For angles between 0° and 45°, the degrees are at the **top** of the page, the minutes in the **left-hand** column, and the functions in the columns designated by the names at the top.

For angles between 45° and 90°, the degrees are at the **foot** of the page, the minutes in the **right-hand** column, and the functions in the columns designated by the names at the **bottom.**

The sines and the cosines of all acute angles, the tangents of angles between 0° and 45°, and the cotangents of angles between 45° and 90° are all less than unity. Hence the characteristics of their logarithms are negative, but the -10 has been omitted in the tables. The true logarithm of all other values are given.

Illustrations.—1. Find log sin 28° 40′ 18″.

From table II, log sin 28° 40′ = 9.68098 and log sin 28° 41′ = 9.68121.

The tabular difference is 23.

Hence, $\frac{18}{60} \times 23$ or .00007 must be added to 9.68098.

$\therefore$ log sin 28° 40′ 18″ = 9.68105.

2. Find log cos 63° 15′ 25″.

From table II, log cos 63 °15′ = 9.65331,
and log cos 63° 16′ = 9.65306.

The tabular difference is 25. Hence, $\frac{25}{60} \times 25$ or .00010 must be subtracted from 9.65331.

$\therefore$ log cos 63° 15′ 25″ = 9.65321.

Note.—*Since the sine and the tangent of an angle increase as the angle increases and the cosine and the cotangent of an angle decrease as the angle increases, the correction in finding* **log sin** *and* **log tan** *must be* **added to,** *and the correction in finding* **log cos** *and* **log cot** *must be* **subtracted** *from the logarithm of the smaller angle.*

3. Given log tan θ = 0.32654, find θ.

From the tables, log tan 64° 45′ = 0.32640,
and log tan 64° 46′ = 0.32673.

Since log tan θ lies between log tan 64° 45′ and log tan 64° 46′, θ must have a value greater than 64° 45′ and less than 64° 46′.

The difference between log tan θ and log tan 64° 45′ is 14. The tabular difference is 33. Hence, $\frac{14}{33}$ of 60″ or 25″ must be added to 64° 45′.

$$\therefore \theta = 64° 45' 25''.$$

Check the following:
log sin 25° 10′ 20″ = 9.62874.
log cos 40° 28′ 35″ = 9.88120.
log tan θ = 9.86432, θ = 36° 11′ 31″.
log cos θ = 9.45826, θ = 73° 18′ 24″.

PROBLEMS

Find the value of the following:
1. log sin 30° 28′ 10″.
2. log cos 42° 42′ 30″.
3. log tan 50° 25′ 45″.
4. log cot 26° 30′ 25″.
5. log sin 81° 10′ 5″.
6. log tan 16° 46′ 18″.

Find θ in the following:
7. log sin θ = 9.35764.
8. log cos θ = 9.67619.
9. log tan θ = 9.98496.
10. log cot θ = 0.40857.

580. **Natural Functions of Angles.**—In §289 a table of Natural Functions of a few angles was given. To find the function of any acute angle, more extended tables must be used.

By consulting the tables in this chapter, it will be found that sin 30° 20° = 0.5050 and tan 48° 37′ = 1.1349. Suppose it is desired to find sin 40° 25′ 20″. Sin 40° 25′ = 0.6483 and sin 40° 26′ = 0.6486. There is a difference of 3 for a difference of 1 minute in the angle. 20″ is $\frac{1}{3}$ of a minute, therefore, take $\frac{1}{3}$ of 3 or 1 and add it to 0.6483. Hence sin 40° 25′ 20″ = 0.6484.

Note.—*Since the sine and the tangent of an angle increase as the angle increases, the fractional part of the difference must be added to the function of the smaller angle. On the other hand since the cosine and the cotangent of an angle decrease as the angle increases, the fractional part of the difference must be subtracted from the functions of the smaller angle.*

Conversely, suppose sin θ = 0.8023 and the angle θ is to be found. By consulting the tables it is found that θ = 53° 21′. If tan θ = 0.3969, θ = 21° 39′. Again, if sin θ = 0.2800, θ = 16° 15′ 40″. Here sin 16° 15′ = 0.2798 and sin 16° 16′ = 0.2801, a difference of 3. The difference between the value of sin 16° 15′ and sin θ is 2. Now, $\frac{2}{3}$ of 60″ or 40″ must be added to 16°15′. Hence, if sin θ = 0.2800, θ = 16° 15′ 40″.

PROBLEMS

1. Find the value of the following:
(*a*) sin 28° 37′.
(*b*) cos 48° 55′.
(*c*) tan 73° 22′.
(*d*) sin 43° 25′ 30″.
(*e*) cos 55° 52′ 45″.
(*f*) tan 23° 45′ 15″.

2. Find the value of θ in the following:
(*a*) tan θ = 0.6847.
(*b*) cos θ = 0.5878.
(*c*) sin θ = 0.9400.
(*d*) sin θ = 0.3463.
(*e*) cos θ = 0.7362.
(*f*) tan θ = 1.7362.

TABLE I

THE
COMMON LOGARITHMS
OF THE
NATURAL NUMBERS

From *Plane Trigonometry With Tables*
By Eugene Henry Barker.

ha

No.	0	1	2	3	4	5	6	7	8	9
100	00 000	00 043	00 087	00 130	00 173	00 217	00 260	00 303	00 346	00 389
101	00 432	00 475	00 518	00 561	00 604	00 647	00 689	00 732	00 775	00 817
102	00 860	00 903	00 945	00 988	01 030	01 072	01 115	01 157	01 199	01 242
103	01 284	01 326	01 368	01 410	01 452	01 494	01 536	01 578	01 620	01 662
104	01 703	01 745	01 787	01 828	01 870	01 912	01 953	01 995	02 036	02 078
105	02 119	02 160	02 202	02 243	02 284	02 325	02 366	02 407	02 449	02 490
106	02 531	02 572	02 612	02 653	02 694	02 735	02 776	02 816	02 857	02 898
107	02 938	02 979	03 019	03 060	03 100	03 141	03 181	03 222	03 262	03 302
108	03 342	03 383	03 423	03 463	03 503	03 543	03 583	03 623	03 663	03 703
109	03 743	03 782	03 822	03 862	03 902	03 941	03 981	04 021	04 060	04 100
110	04 139	04 179	04 218	04 258	04 297	04 336	04 376	04 415	04 454	04 493
111	04 532	04 571	04 610	04 650	04 689	04 727	04 766	04 805	04 844	04 883
112	04 922	04 961	04 999	05 038	05 077	05 115	05 154	05 192	05 231	05 269
113	05 308	05 346	05 385	05 423	05 461	05 500	05 538	05 576	05 614	05 652
114	05 690	05 729	05 767	05 805	05 843	05 881	05 918	05 956	05 994	06 032
115	06 070	06 108	06 145	06 183	06 221	06 258	06 296	06 333	06 371	06 408
116	06 446	06 483	06 521	06 558	06 595	06 633	06 670	06 707	06 744	06 781
117	06 819	06 856	06 893	06 930	06 967	07 004	07 041	07 078	07 115	07 151
118	07 188	07 225	07 262	07 298	07 335	07 372	07 408	07 445	07 482	07 518
119	07 555	07 591	07 628	07 664	07 700	07 737	07 773	07 809	07 846	07 882
120	07 918	07 954	07 990	08 027	08 063	08 099	08 135	08 171	08 207	08 243
121	08 279	08 314	08 350	08 386	08 422	08 458	08 493	08 529	08 565	08 600
122	08 636	08 672	08 707	08 743	08 778	08 814	08 849	08 884	08 920	08 955
123	08 991	09 026	09 061	09 096	09 132	09 167	09 202	09 237	09 272	09 307
124	09 342	09 377	09 412	09 447	09 482	09 517	09 552	09 587	09 621	09 656
125	09 691	09 726	09 760	09 795	09 830	09 864	09 899	09 934	09 968	10 003
126	10 037	10 072	10 106	10 140	10 175	10 209	10 243	10 278	10 312	10 346
127	10 380	10 415	10 449	10 483	10 517	10 551	10 585	10 619	10 653	10 687
128	10 721	10 755	10 789	10 823	10 857	10 890	10 924	10 958	10 992	11 025
129	11 059	11 093	11 126	11 160	11 193	11 227	11 261	11 294	11 327	11 361
130	11 394	11 428	11 461	11 494	11 528	11 561	11 594	11 628	11 661	11 694
131	11 727	11 760	11 793	11 826	11 860	11 893	11 926	11 959	11 992	12 024
132	12 057	12 090	12 123	12 156	12 189	12 222	12 254	12 287	12 320	12 352
133	12 385	12 418	12 450	12 483	12 516	12 548	12 581	12 613	12 646	12 678
134	12 710	12 743	12 775	12 808	12 840	12 872	12 905	12 937	12 969	13 001
135	13 033	13 066	13 098	13 130	13 162	13 194	13 226	13 258	13 290	13 322
136	13 354	13 386	13 418	13 450	13 481	13 513	13 545	13 577	13 609	13 640
137	13 672	13 704	13 735	13 767	13 799	13 830	13 862	13 893	13 925	13 956
138	13 988	14 019	14 051	14 082	14 114	14 145	14 176	14 208	14 239	14 270
139	14 301	14 333	14 364	14 395	14 426	14 457	14 489	14 520	14 551	14 582
140	14 613	14 644	14 675	14 706	14 737	14 768	14 799	14 829	14 860	14 891
141	14 922	14 953	14 983	15 014	15 045	15 076	15 106	15 137	15 168	15 198
142	15 229	15 259	15 290	15 320	15 351	15 381	15 412	15 442	15 473	15 503
143	15 534	15 564	15 594	15 625	15 655	15 685	15 715	15 746	15 776	15 806
144	15 836	15 866	15 897	15 927	15 957	15 987	16 017	16 047	16 077	16 107
145	16 137	16 167	16 197	16 227	16 256	16 286	16 316	16 346	16 376	16 406
146	16 435	16 465	16 495	16 524	16 554	16 584	16 613	16 643	16 673	16 702
147	16 732	16 761	16 791	16 820	16 850	16 879	16 909	16 938	16 967	16 997
148	17 026	17 056	17 085	17 114	17 143	17 173	17 202	17 231	17 260	17 289
149	17 319	17 348	17 377	17 406	17 435	17 464	17 493	17 522	17 551	17 580
150	17 609	17 638	17 667	17 696	17 725	17 754	17 782	17 811	17 840	17 869

No.	0	1	2	3	4	5	6	7	8	9
150	17 609	17 638	17 667	17 696	17 725	17 754	17 782	17 811	17 840	17 869
151	17 898	17 926	17 955	17 984	18 013	18 041	18 070	18 099	18 127	18 156
152	18 184	18 213	18 241	18 270	18 298	18 327	18 355	18 384	18 412	18 441
153	18 469	18 498	18 526	18 554	18 583	18 611	18 639	18 667	18 696	18 724
154	18 752	18 780	18 808	18 837	18 865	18 893	18 921	18 949	18 977	19 005
155	19 033	19 061	19 089	19 117	19 145	19 173	19 201	19 229	19 257	19 285
156	19 312	19 340	19 368	19 396	19 424	19 451	19 479	19 507	19 535	19 562
157	19 590	19 618	19 645	19 673	19 700	19 728	19 756	19 783	19 811	19 838
158	19 866	19 893	19 921	19 948	19 976	20 003	20 030	20 058	20 085	20 112
159	20 140	20 167	20 194	20 222	20 249	20 276	20 303	20 330	20 358	20 385
160	20 412	20 439	20 466	20 493	20 520	20 548	20 575	20 602	20 629	20 656
161	20 683	20 710	20 737	20 763	20 790	20 817	20 844	20 871	20 898	20 925
162	20 952	20 978	21 005	21 032	21 059	21 085	21 112	21 139	21 165	21 192
163	21 219	21 245	21 272	21 299	21 325	21 352	21 378	21 405	21 431	21 458
164	21 484	21 511	21 537	21 564	21 590	21 617	21 643	21 669	21 696	21 722
165	21 748	21 775	21 801	21 827	21 854	21 880	21 906	21 932	21 958	21 985
166	22 011	22 037	22 063	22 089	22 115	22 141	22 167	22 194	22 220	22 246
167	22 272	22 298	22 324	22 350	22 376	22 401	22 427	22 453	22 479	22 505
168	22 531	22 557	22 583	22 608	22 634	22 660	22 686	22 712	22 737	22 763
169	22 789	22 814	22 840	22 866	22 891	22 917	22 943	22 968	22 994	23 019
170	23 045	23 070	23 096	23 121	23 147	23 172	23 198	23 223	23 249	23 274
171	23 300	23 325	23 350	23 376	23 401	23 426	23 452	23 477	23 502	23 528
172	23 553	23 578	23 603	23 629	23 654	23 679	23 704	23 729	23 754	23 779
173	23 805	23 830	23 855	23 880	23 905	23 930	23 955	23 980	24 005	24 030
174	24 055	24 080	24 105	24 130	24 155	24 180	24 204	24 229	24 254	24 279
175	24 304	24 329	24 353	24 378	24 403	24 428	24 452	24 477	24 502	24 527
176	24 551	24 576	24 601	24 625	24 650	24 674	24 699	24 724	24 748	24 773
177	24 797	24 822	24 846	24 871	24 895	24 920	24 944	24 969	24 993	25 018
178	25 042	25 066	25 091	25 115	25 139	25 164	25 188	25 212	25 237	25 261
179	25 285	25 310	25 334	25 358	25 382	25 406	25 431	25 455	25 479	25 503
180	25 527	25 551	25 575	25 600	25 624	25 648	25 672	25 696	25 720	25 744
181	25 768	25 792	25 816	25 840	25 864	25 888	25 912	25 935	25 959	25 983
182	26 007	26 031	26 055	26 079	26 102	26 126	26 150	26 174	26 198	26 221
183	26 245	26 269	26 293	26 316	26 340	26 364	26 387	26 411	26 435	26 458
184	26 482	26 505	26 529	26 553	26 576	26 600	26 623	26 647	26 670	26 694
185	26 717	26 741	26 764	26 788	26 811	26 834	26 858	26 881	26 905	26 928
186	26 951	26 975	26 998	27 021	27 045	27 068	27 091	27 114	27 138	27 161
187	27 184	27 207	27 231	27 254	27 277	27 300	27 323	27 346	27 370	27 393
188	27 416	27 439	27 462	27 485	27 508	27 531	27 554	27 577	27 600	27 623
189	27 646	27 669	27 692	27 715	27 738	27 761	27 784	27 807	27 830	27 852
190	27 875	27 898	27 921	27 944	27 967	27 989	28 012	28 035	28 058	28 081
191	28 103	28 126	28 149	28 171	28 194	28 217	28 240	28 262	28 285	28 307
192	28 330	28 353	28 375	28 398	28 421	28 443	28 466	28 488	28 511	28 533
193	28 556	28 578	28 601	28 623	28 646	28 668	28 691	28 713	28 735	28 758
194	28 780	28 803	28 825	28 847	28 870	28 892	28 914	28 937	28 959	28 981
195	29 003	29 026	29 048	29 070	29 092	29 115	29 137	29 159	29 181	29 203
196	29 226	29 248	29 270	29 292	29 314	29 336	29 358	29 380	29 403	29 425
197	29 447	29 469	29 491	29 513	29 535	29 557	29 579	29 601	29 623	29 645
198	29 667	29 688	29 710	29 732	29 754	29 776	29 798	29 820	29 842	29 863
199	29 885	29 907	29 929	29 951	29 973	29 994	30 016	30 038	30 060	30 081
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298

No.	0	1	2	3	4	5	6	7	8	9
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298
201	30 320	30 341	30 363	30 384	30 406	30 428	30 449	30 471	30 492	30 514
202	30 535	30 557	30 578	30 600	30 621	30 643	30 664	30 685	30 707	30 728
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230	36 173	36 192	36 211	36 229	36 248	36 267	36 286	36 305	36 324	36 342
231	36 361	36 380	36 399	36 418	36 436	36 455	36 474	36 493	36 511	36 530
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237	37 475	37 493	37 511	37 530	37 548	37 566	37 585	37 603	37 621	37 639
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247	39 270	39 287	39 305	39 322	39 340	39 358	39 375	39 393	39 410	39 428
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249	39 620	39 637	39 655	39 672	39 690	39 707	39 724	39 742	39 759	39 777
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253	40 312	40 329	40 346	40 364	40 381	40 398	40 415	40 432	40 449	40 466
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258	41 162	41 179	41 196	41 212	41 229	41 246	41 263	41 280	41 296	41 313
259	41 330	41 347	41 363	41 380	41 397	41 414	41 430	41 447	41 464	41 481
260	41 497	41 514	41 531	41 547	41 564	41 581	41 597	41 614	41 631	41 647
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265	42 325	42 341	42 357	42 374	42 390	42 406	42 423	42 439	42 455	42 472
266	42 488	42 504	42 521	42 537	42 553	42 570	42 586	42 602	42 619	42 635
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277	44 248	44 264	44 279	44 295	44 311	44 326	44 342	44 358	44 373	44 389
278	44 404	44 420	44 436	44 451	44 467	44 483	44 498	44 514	44 529	44 545
279	44 560	44 576	44 592	44 607	44 623	44 638	44 654	44 669	44 685	44 700
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283	45 179	45 194	45 209	45 225	45 240	45 255	45 271	45 286	45 301	45 317
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289	46 090	46 105	46 120	46 135	46 150	46 165	46 180	46 195	46 210	46 225
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292	46 538	46 553	46 568	46 583	46 598	46 613	46 627	46 642	46 657	46 672
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311	49 276	49 290	49 304	49 318	49 332	49 346	49 360	49 374	49 388	49 402
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373	57 171	57 183	57 194	57 206	57 217	57 229	57 241	57 252	57 264	57 276
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409	61 172	61 183	61 194	61 204	61 215	61 225	61 236	61 247	61 257	61 268
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417	62 014	62 024	62 034	62 045	62 055	62 066	62 076	62 086	62 097	62 107
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421	62 428	62 439	62 449	62 459	62 469	62 480	62 490	62 500	62 511	62 521
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423	62 634	62 644	62 655	62 665	62 675	62 685	62 696	62 706	62 716	62 726
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438	64 147	64 157	64 167	64 177	64 187	64 197	64 207	64 217	64 227	64 237
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447	65 031	65 040	65 050	65 060	65 070	65 079	65 089	65 099	65 108	65 118
448	65 128	65 137	65 147	65 157	65 167	65 176	65 186	65 196	65 205	65 215
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450	65 321	65 331	65 341	65 350	65 360	65 369	65 379	65 389	65 398	65 408

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456	65 896	65 906	65 916	65 925	65 935	65 944	65 954	65 963	65 973	65 982
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489	68 931	68 940	68 949	68 958	68 966	68 975	68 984	68 993	69 002	69 011
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571	75 664	75 671	75 679	75 686	75 694	75 702	75 709	75 717	75 724	75 732
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617	79 029	79 036	79 043	79 050	79 057	79 064	79 071	79 078	79 085	79 092
618	79 099	79 106	79 113	79 120	79 127	79 134	79 141	79 148	79 155	79 162
619	79 169	79 176	79 183	79 190	79 197	79 204	79 211	79 218	79 225	79 232
620	79 239	79 246	79 253	79 260	79 267	79 274	79 281	79 288	79 295	79 302
621	79 309	79 316	79 323	79 330	79 337	79 344	79 351	79 358	79 365	79 372
622	79 379	79 386	79 393	79 400	79 407	79 414	79 421	79 428	79 435	79 442
623	79 449	79 456	79 463	79 470	79 477	79 484	79 491	79 498	79 505	79 511
624	79 518	79 525	79 532	79 539	79 546	79 553	79 560	79 567	79 574	79 581
625	79 588	79 595	79 602	79 609	79 616	79 623	79 630	79 637	79 644	79 650
626	79 657	79 664	79 671	79 678	79 685	79 692	79 699	79 706	79 713	79 720
627	79 727	79 734	79 741	79 748	79 754	79 761	79 768	79 775	79 782	79 789
628	79 796	79 803	79 810	79 817	79 824	79 831	79 837	79 844	79 851	79 858
629	79 865	79 872	79 879	79 886	79 893	79 900	79 906	79 913	79 920	79 927
630	79 934	79 941	79 948	79 955	79 962	79 969	79 975	79 982	79 989	79 996
631	80 003	80 010	80 017	80 024	80 030	80 037	80 044	80 051	80 058	80 065
632	80 072	80 079	80 085	80 092	80 099	80 106	80 113	80 120	80 127	80 134
633	80 140	80 147	80 154	80 161	80 168	80 175	80 182	80 188	80 195	80 202
634	80 209	80 216	80 223	80 229	80 236	80 243	80 250	80 257	80 264	80 271
635	80 277	80 284	80 291	80 298	80 305	80 312	80 318	80 325	80 332	80 339
636	80 346	80 353	80 359	80 366	80 373	80 380	80 387	80 393	80 400	80 407
637	80 414	80 421	80 428	80 434	80 441	80 448	80 455	80 462	80 468	80 475
638	80 482	80 489	80 496	80 502	80 509	80 516	80 523	80 530	80 536	80 543
639	80 550	80 557	80 564	80 570	80 577	80 584	80 591	80 598	80 604	80 611
640	80 618	80 625	80 632	80 638	80 645	80 652	80 659	80 665	80 672	80 679
641	80 686	80 693	80 699	80 706	80 713	80 720	80 726	80 733	80 740	80 747
642	80 754	80 760	80 767	80 774	80 781	80 787	80 794	80 801	80 808	80 814
643	80 821	80 828	80 835	80 841	80 848	80 855	80 862	80 868	80 875	80 882
644	80 889	80 895	80 902	80 909	80 916	80 922	80 929	80 936	80 943	80 949
645	80 956	80 963	80 969	80 976	80 983	80 990	80 996	81 003	81 010	81 017
646	81 023	81 030	81 037	81 043	81 050	81 057	81 064	81 070	81 077	81 084
647	81 090	81 097	81 104	81 111	81 117	81 124	81 131	81 137	81 144	81 151
648	81 158	81 164	81 171	81 178	81 184	81 191	81 198	81 204	81 211	81 218
649	81 224	81 231	81 238	81 245	81 251	81 258	81 265	81 271	81 278	81 285
650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351

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650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351
651	81 358	81 365	81 371	81 378	81 385	81 391	81 398	81 405	81 411	81 418
652	81 425	81 431	81 438	81 445	81 451	81 458	81 465	81 471	81 478	81 485
653	81 491	81 498	81 505	81 511	81 518	81 525	81 531	81 538	81 544	81 551
654	81 558	81 564	81 571	81 578	81 584	81 591	81 598	81 604	81 611	81 617
655	81 624	81 631	81 637	81 644	81 651	81 657	81 664	81 671	81 677	81 684
656	81 690	81 697	81 704	81 710	81 717	81 723	81 730	81 737	81 743	81 750
657	81 757	81 763	81 770	81 776	81 783	81 790	81 796	81 803	81 809	81 816
658	81 823	81 829	81 836	81 842	81 849	81 856	81 862	81 869	81 875	81 882
659	81 889	81 895	81 902	81 908	81 915	81 921	81 928	81 935	81 941	81 948
660	81 954	81 961	81 968	81 974	81 981	81 987	81 994	82 000	82 007	82 014
661	82 020	82 027	82 033	82 040	82 046	82 053	82 060	82 066	82 073	82 079
662	82 086	82 092	82 099	82 105	82 112	82 119	82 125	82 132	82 138	82 145
663	82 151	82 158	82 164	82 171	82 178	82 184	82 191	82 197	82 204	82 210
664	82 217	82 223	82 230	82 236	82 243	82 249	82 256	82 263	82 269	82 276
665	82 282	82 289	82 295	82 302	82 308	82 315	82 321	82 328	82 334	82 341
666	82 347	82 354	82 360	82 367	82 373	82 380	82 387	82 393	82 400	82 406
667	82 413	82 419	82 426	82 432	82 439	82 445	82 452	82 458	82 465	82 471
668	82 478	82 484	82 491	82 497	82 504	82 510	82 517	82 523	82 530	82 536
669	82 543	82 549	82 556	82 562	82 569	82 575	82 582	82 588	82 595	82 601
670	82 607	82 614	82 620	82 627	82 633	82 640	82 646	82 653	82 659	82 666
671	82 672	82 679	82 685	82 692	82 698	82 705	82 711	82 718	82 724	82 730
672	82 737	82 743	82 750	82 756	82 763	82 769	82 776	82 782	82 789	82 795
673	82 802	82 808	82 814	82 821	82 827	82 834	82 840	82 847	82 853	82 860
674	82 866	82 872	82 879	82 885	82 892	82 898	82 905	82 911	82 918	82 924
675	82 930	82 937	82 943	82 950	82 956	82 963	82 969	82 975	82 982	82 988
676	82 995	83 001	83 008	83 014	83 020	83 027	83 033	83 040	83 046	83 052
677	83 059	83 065	83 072	83 078	83 085	83 091	83 097	83 104	83 110	83 117
678	83 123	83 129	83 136	83 142	83 149	83 155	83 161	83 168	83 174	83 181
679	83 187	83 193	83 200	83 206	83 213	83 219	83 225	83 232	83 238	83 245
680	83 251	83 257	83 264	83 270	83 276	83 283	83 289	83 296	83 302	83 308
681	83 315	83 321	83 327	83 334	83 340	83 347	83 353	83 359	83 366	83 372
682	83 378	83 385	83 391	83 398	83 404	83 410	83 417	83 423	83 429	83 436
683	83 442	83 448	83 455	83 461	83 467	83 474	83 480	83 487	83 493	83 499
684	83 506	83 512	83 518	83 525	83 531	83 537	83 544	83 550	83 556	83 563
685	83 569	83 575	83 582	83 588	83 594	83 601	83 607	83 613	83 620	83 626
686	83 632	83 639	83 645	83 651	83 658	83 664	83 670	83 677	83 683	83 689
687	83 696	83 702	83 708	83 715	83 721	83 727	83 734	83 740	83 746	83 753
688	83 759	83 765	83 771	83 778	83 784	83 790	83 797	83 803	83 809	83 816
689	83 822	83 828	83 835	83 841	83 847	83 853	83 860	83 866	83 872	83 879
690	83 885	83 891	83 897	83 904	83 910	83 916	83 923	83 929	83 935	83 942
691	83 948	83 954	83 960	83 967	83 973	83 979	83 985	83 992	83 998	84 004
692	84 011	84 017	84 023	84 029	84 036	84 042	84 048	84 055	84 061	84 067
693	84 073	84 080	84 086	84 092	84 098	84 105	84 111	84 117	84 123	84 130
694	84 136	84 142	84 148	84 155	84 161	84 167	84 173	84 180	84 186	84 192
695	84 198	84 205	84 211	84 217	84 223	84 230	84 236	84 242	84 248	84 255
696	84 261	84 267	84 273	84 280	84 286	84 292	84 298	84 305	84 311	84 317
697	84 323	84 330	84 336	84 342	84 348	84 354	84 361	84 367	84 373	84 379
698	84 386	84 392	84 398	84 404	84 410	84 417	84 423	84 429	84 435	84 442
699	84 448	84 454	84 460	84 466	84 473	84 479	84 485	84 491	84 497	84 504
700	84 510	84 516	84 522	84 528	84 535	84 541	84 547	84 553	84 559	84 566

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700	84 510	84 516	84 522	84 528	84 535	84 541	84 547	84 553	84 559	84 566
701	84 572	84 578	84 584	84 590	84 597	84 603	84 609	84 615	84 621	84 628
702	84 634	84 640	84 646	84 652	84 658	84 665	84 671	84 677	84 683	84 689
703	84 696	84 702	84 708	84 714	84 720	84 726	84 733	84 739	84 745	84 751
704	84 757	84 763	84 770	84 776	84 782	84 788	84 794	84 800	84 807	84 813
705	84 819	84 825	84 831	84 837	84 844	84 850	84 856	84 862	84 868	84 874
706	84 880	84 887	84 893	84 899	84 905	84 911	84 917	84 924	84 930	84 936
707	84 942	84 948	84 954	84 960	84 967	84 973	84 979	84 985	84 991	84 997
708	85 003	85 009	85 016	85 022	85 028	85 034	85 040	85 046	85 052	85 058
709	85 065	85 071	85 077	85 083	85 089	85 095	85 101	85 107	85 114	85 120
710	85 126	85 132	85 138	85 144	85 150	85 156	85 163	85 169	85 175	85 181
711	85 187	85 193	85 199	85 205	85 211	85 217	85 224	85 230	85 236	85 242
712	85 248	85 254	85 260	85 266	85 272	85 278	85 285	85 291	85 297	85 303
713	85 309	85 315	85 321	85 327	85 333	85 339	85 345	85 352	85 358	85 364
714	85 370	85 376	85 382	85 388	85 394	85 400	85 406	85 412	85 418	85 425
715	85 431	85 437	85 443	85 449	85 455	85 461	85 467	85 473	85 479	85 485
716	85 491	85 497	85 503	85 509	85 516	85 522	85 528	85 534	85 540	85 546
717	85 552	85 558	85 564	85 570	85 576	85 582	85 588	85 594	85 600	85 606
718	85 612	85 618	85 625	85 631	85 637	85 643	85 649	85 655	85 661	85 667
719	85 673	85 679	85 685	85 691	85 697	85 703	85 709	85 715	85 721	85 727
720	85 733	85 739	85 745	85 751	85 757	85 763	85 769	85 775	85 781	85 788
721	85 794	85 800	85 806	85 812	85 818	85 824	85 830	85 836	85 842	85 848
722	85 854	85 860	85 866	85 872	85 878	85 884	85 890	85 896	85 902	85 908
723	85 914	85 920	85 926	85 932	85 938	85 944	85 950	85 956	85 962	85 968
724	85 974	85 980	85 986	85 992	85 998	86 004	86 010	86 016	86 022	86 028
725	86 034	86 040	86 046	86 052	86 058	86 064	86 070	86 076	86 082	86 088
726	86 094	86 100	86 106	86 112	86 118	86 124	86 130	86 136	86 141	86 147
727	86 153	86 159	86 165	86 171	86 177	86 183	86 189	86 195	86 201	86 207
728	86 213	86 219	86 225	86 231	86 237	86 243	86 249	86 255	86 261	86 267
729	86 273	86 279	86 285	86 291	86 297	86 303	86 308	86 314	86 320	86 326
730	86 332	86 338	86 344	86 350	86 356	86 362	86 368	86 374	86 380	86 386
731	86 392	86 398	86 404	86 410	86 415	86 421	86 427	86 433	86 439	86 445
732	86 451	86 457	86 463	86 469	86 475	86 481	86 487	86 493	86 499	86 504
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734	86 570	86 576	86 581	86 587	86 593	86 599	86 605	86 611	86 617	86 623
735	86 629	86 635	86 641	86 646	86 652	86 658	86 664	86 670	86 676	86 682
736	86 688	86 694	86 700	86 705	86 711	86 717	86 723	86 729	86 735	86 741
737	86 747	86 753	86 759	86 764	86 770	86 776	86 782	86 788	86 794	86 800
738	86 806	86 812	86 817	86 823	86 829	86 835	86 841	86 847	86 853	85 859
739	86 864	86 870	86 876	86 882	86 888	86 894	86 900	86 906	86 911	86 917
740	86 923	86 929	86 935	86 941	86 947	86 953	86 958	86 964	86 970	86 976
741	86 982	86 988	86 994	86 999	87 005	87 011	87 017	87 023	87 029	87 035
742	87 040	87 046	87 052	87 058	87 064	87 070	87 075	87 081	87 087	87 093
743	87 099	87 105	87 111	87 116	87 122	87 128	87 134	87 140	87 146	87 151
744	87 157	87 163	87 169	87 175	87 181	87 186	87 192	87 198	87 204	87 210
745	87 216	87 221	87 227	87 233	87 239	87 245	87 251	87 256	87 262	87 268
746	87 274	87 280	87 286	87 291	87 297	87 303	87 309	87 315	87 320	87 326
747	87 332	87 338	87 344	87 349	87 355	87 361	87 367	87 373	87 379	87 384
748	87 390	87 396	87 402	87 408	87 413	87 419	87 425	87 431	87 437	87 442
749	87 448	87 454	87 460	87 466	87 471	87 477	87 483	87 489	87 495	87 500
750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558

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750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558
751	87 564	87 570	87 576	87 581	87 587	87 593	87 599	87 604	87 610	87 616
752	87 622	87 628	87 633	87 639	87 645	87 651	87 656	87 662	87 668	87 674
753	87 679	87 685	87 691	87 697	87 703	87 708	87 714	87 720	87 726	87 731
754	87 737	87 743	87 749	87 754	87 760	76 766	87 772	87 777	87 783	87 789
755	87 795	87 800	87 806	87 812	87 818	87 823	87 829	87 835	87 841	87 846
756	87 852	87 858	87 864	87 869	87 875	87 881	87 887	87 892	87 898	87 904
757	87 910	87 915	87 921	87 927	87 933	87 938	87 944	87 950	87 955	87 961
758	87 967	87 973	87 978	87 984	87 990	87 996	88 001	88 007	88 013	88 018
759	88 024	88 030	88 036	88 041	88 047	88 053	88 058	88 064	88 070	88 076
760	88 081	88 087	88 093	88 098	88 104	88 110	88 116	88 121	88 127	88 133
761	88 138	88 144	88 150	88 156	88 161	88 167	88 173	88 178	88 184	88 190
762	88 195	88 201	88 207	88 213	88 218	88 224	88 230	88 235	88 241	88 247
763	88 252	88 258	88 264	88 270	88 275	88 281	88 287	88 292	88 298	88 304
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765	88 366	88 372	88 377	88 383	88 389	88 395	88 400	88 406	88 412	88 417
766	88 423	88 429	88 434	88 440	88 446	88 451	88 457	88 463	88 468	88 474
767	88 480	88 485	88 491	88 497	88 502	88 508	88 513	88 519	88 525	88 530
768	88 536	88 542	88 547	88 553	88 559	88 564	88 570	88 576	88 581	88 587
769	88 593	88 598	88 604	88 610	88 615	88 621	88 627	88 632	88 638	88 643
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771	88 705	88 711	88 717	88 722	88 728	88 734	88 739	88 745	88 750	88 756
772	88 762	88 767	88 773	88 779	88 784	88 790	88 795	88 801	88 807	88 812
773	88 818	88 824	88 829	88 835	88 840	88 846	88 852	88 857	88 863	88 868
774	88 874	88 880	88 885	88 891	88 897	88 902	88 908	88 913	88 919	88 925
775	88 930	88 936	88 941	88 947	88 953	88 958	88 964	88 969	88 975	88 981
776	88 986	88 992	88 997	89 003	89 009	89 014	89 020	89 025	89 031	89 037
777	89 042	89 048	89 053	89 059	89 064	89 070	89 076	89 081	89 087	89 092
778	89 098	89 104	89 109	89 115	89 120	89 126	89 131	89 137	89 143	89 148
779	89 154	89 159	89 165	89 170	89 176	89 182	89 187	89 193	89 198	89 204
780	89 209	89 215	89 221	89 226	89 232	89 237	89 243	89 248	89 254	89 260
781	89 265	89 271	89 276	89 282	89 287	89 293	89 298	89 304	89 310	89 315
782	89 321	89 326	89 332	89 337	89 343	89 348	89 354	89 360	89 365	89 371
783	89 376	89 382	89 387	89 393	89 398	89 404	89 409	89 415	89 421	89 426
784	89 432	89 437	89 443	89 448	89 454	89 459	89 465	89 470	89 476	89 481
785	89 487	89 492	89 498	89 504	89 509	89 515	89 520	89 526	89 531	89 537
786	89 542	89 548	89 553	89 559	89 564	89 570	89 575	89 581	89 586	89 592
787	89 597	89 603	89 609	89 614	89 620	89 625	89 631	89 636	89 642	89 647
788	89 653	89 658	89 664	89 669	89 675	89 680	89 686	89 691	89 697	89 702
789	89 708	89 713	89 719	89 724	89 730	89 735	89 741	89 746	89 752	89 757
790	89 763	89 768	89 774	89 779	89 785	89 790	89 796	89 801	89 807	89 812
791	89 818	89 823	89 829	89 834	89 840	89 845	89 851	89 856	89 862	89 867
792	89 873	89 878	89 883	89 889	89 894	89 900	89 905	89 911	89 916	89 922
793	89 927	89 933	89 938	89 944	89 949	89 955	89 960	89 966	89 971	89 977
794	89 982	89 988	89 993	89 998	90 004	90 009	90 015	90 020	90 026	90 031
795	90 037	90 042	90 048	90 053	90 059	90 064	90 069	90 075	90 080	90 086
796	90 091	90 097	90 102	90 108	90 113	90 119	90 124	90 129	90 135	90 140
797	90 146	90 151	90 157	90 162	90 168	90 173	90 179	90 184	90 189	90 195
798	90 200	90 206	90 211	90 217	90 222	90 227	90 233	90 238	90 244	90 249
799	90 255	90 260	90 266	90 271	90 276	90 282	90 287	90 293	90 298	90 304
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358

No.	0	1	2	3	4	5	6	7	8	9
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358
801	90 363	90 369	90 374	90 380	90 385	90 390	90 396	90 401	90 407	90 412
802	90 417	90 423	90 428	90 434	90 439	90 445	90 450	90 455	90 461	90 466
803	90 472	90 477	90 482	90 488	90 493	90 499	90 504	90 509	90 515	90 520
804	90 526	90 531	90 536	90 542	90 547	90 553	90 558	90 563	90 569	90 574
805	90 580	90 585	90 590	90 596	90 601	90 607	90 612	90 617	90 623	90 628
806	90 634	90 639	90 644	90 650	90 655	90 660	90 666	90 671	90 677	90 682
807	90 687	90 693	90 698	90 703	90 709	90 714	90 720	90 725	90 730	90 736
808	90 741	90 747	90 752	90 757	90 763	90 768	90 773	90 779	90 784	90 789
809	90 795	90 800	90 806	90 811	90 816	90 822	90 827	90 832	90 838	90 843
810	90 849	90 854	90 859	90 865	90 870	90 875	90 881	90 886	90 891	90 897
811	90 902	90 907	90 913	90 918	90 924	90 929	90 934	90 940	90 945	90 950
812	90 956	90 961	90 966	90 972	90 977	90 982	90 988	90 993	90 998	91 004
813	91 009	91 014	91 020	91 025	91 030	91 036	91 041	91 046	91 052	91 057
814	91 062	91 068	91 073	91 078	91 084	91 089	91 094	91 100	91 105	91 110
815	91 116	91 121	91 126	91 132	91 137	91 142	91 148	91 153	91 158	91 164
816	91 169	91 174	91 180	91 185	91 190	91 196	91 201	91 206	91 212	91 217
817	91 222	91 228	91 233	91 238	91 243	91 249	91 254	91 259	91 265	91 270
818	91 275	91 281	91 286	91 291	91 297	91 302	91 307	91 312	91 318	91 323
819	91 328	91 334	91 339	91 344	91 350	91 355	91 360	91 365	91 371	91 376
820	91 381	91 387	91 392	91 397	91 403	91 408	91 413	91 418	91 424	91 429
821	91 434	91 440	91 445	91 450	91 455	91 461	91 466	91 471	91 477	91 482
822	91 487	91 492	91 498	91 503	91 508	91 514	91 519	91 524	91 529	91 535
823	91 540	91 545	91 551	91 556	91 561	91 566	91 572	91 577	91 582	91 587
824	91 593	91 598	91 603	91 609	91 614	91 619	91 624	91 630	91 635	91 640
825	91 645	91 651	91 656	91 661	91 666	91 672	91 677	91 682	91 687	91 693
826	91 698	91 703	91 709	91 714	91 719	91 724	91 730	91 735	91 740	91 745
827	91 751	91 756	91 761	91 766	91 772	91 777	91 782	91 787	91 793	91 798
828	91 803	91 808	91 814	91 819	91 824	91 829	91 834	91 840	91 845	91 850
829	91 855	91 861	91 866	91 871	91 876	91 882	91 887	91 892	91 897	91 903
830	91 908	91 913	91 918	91 924	91 929	91 934	91 939	91 944	91 950	91 955
831	91 960	91 965	91 971	91 976	91 981	91 986	91 991	91 997	92 002	92 007
832	92 012	92 018	92 023	92 028	92 033	92 038	92 044	92 049	92 054	92 059
833	92 065	92 070	92 075	92 080	92 085	92 091	92 096	92 101	92 106	92 111
834	92 117	92 122	92 127	92 132	92 137	92 143	92 148	92 153	92 158	92 163
835	92 169	92 174	92 179	92 184	92 189	92 195	92 200	92 205	92 210	92 215
836	92 221	92 226	92 231	92 236	92 241	92 247	92 252	92 257	92 262	92 267
837	92 273	92 278	92 283	92 288	92 293	92 298	92 304	92 309	92 314	92 319
838	92 324	92 330	92 335	92 340	92 345	92 350	92 355	92 361	92 366	92 371
839	92 376	92 381	93 387	92 392	92 397	92 402	92 407	92 412	92 418	92 423
840	92 428	92 433	92 438	92 443	92 449	92 454	92 459	92 464	92 469	92 474
841	92 480	92 485	92 490	92 495	92 500	92 505	92 511	92 516	92 521	92 526
842	92 531	92 536	92 542	92 547	92 552	92 557	92 562	92 567	92 572	92 578
843	92 583	92 588	92 593	92 598	92 603	92 609	92 614	92 619	92 624	92 629
844	92 634	92 639	92 645	92 650	92 655	92 660	92 665	92 670	92 675	92 681
845	92 686	92 691	92 696	92 701	92 706	92 711	92 716	92 722	92 727	92 732
846	92 737	92 742	92 747	92 752	92 758	92 763	92 768	92 773	92 778	92 783
847	92 788	92 793	92 799	92 804	92 809	92 814	92 819	92 824	92 829	92 834
848	92 840	92 845	92 850	92 855	92 860	92 865	92 870	92 875	92 881	92 886
849	92 891	92 896	92 901	92 906	92 911	92 916	92 921	92 927	92 932	92 937
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988

No.	0	1	2	3	4	5	6	7	8	9
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988
851	92 993	92 998	93 003	93 008	93 013	93 018	93 024	93 029	93 034	93 039
852	93 044	93 049	93 054	93 059	93 064	93 069	93 075	93 080	93 085	93 090
853	93 095	93 100	93 105	93 110	93 115	93 120	93 125	93 131	93 136	93 141
854	93 146	93 151	93 156	93 161	93 166	93 171	93 176	93 181	93 186	93 192
855	93 197	93 202	93 207	93 212	93 217	93 222	93 227	93 232	93 237	93 242
856	93 247	93 252	93 258	93 263	93 268	93 273	93 278	93 283	93 288	93 293
857	93 298	93 303	93 308	93 313	93 318	93 323	93 328	93 334	93 339	93 344
858	93 349	93 354	93 359	93 364	93 369	93 374	93 379	93 384	93 389	93 394
859	93 399	93 404	93 409	93 414	93 420	93 425	93 430	93 435	93 440	93 445
860	93 450	93 455	93 460	93 465	93 470	93 475	93 480	93 485	93 490	93 495
861	93 500	93 505	93 510	93 515	93 520	93 526	93 531	93 536	93 541	93 546
862	93 551	93 556	93 561	93 566	93 571	93 576	93 581	93 586	93 591	93 596
863	93 601	93 606	93 611	93 616	93 621	93 626	93 631	93 636	93 641	93 646
864	93 651	93 656	93 661	93 666	93 671	93 676	93 682	93 687	93 692	93 697
865	93 702	93 707	93 712	93 717	93 722	93 727	93 732	93 737	93 742	93 747
866	93 752	93 757	93 762	93 767	93 772	93 777	93 782	93 787	93 792	93 797
867	93 802	93 807	93 812	93 817	93 822	93 827	93 832	93 837	93 842	93 847
868	93 852	93 857	93 862	93 867	93 872	93 877	93 882	93 887	93 892	93 897
869	93 902	93 907	93 912	93 917	93 922	93 927	93 932	93 937	93 942	93 947
870	93 952	93 957	93 962	93 967	93 972	93 977	93 982	93 987	93 992	93 997
871	94 002	94 007	94 012	94 017	94 022	94 027	94 032	94 037	94 042	94 047
872	94 052	94 057	94 062	94 067	94 072	94 077	94 082	94 086	94 091	94 096
873	94 101	94 106	94 111	94 116	94 121	94 126	94 131	94 136	94 141	94 146
874	94 151	94 156	94 161	94 166	94 171	94 176	94 181	94 186	94 191	94 196
875	94 201	94 206	94 211	94 216	94 221	94 226	94 231	94 236	94 240	94 245
876	94 250	94 255	94 260	94 265	94 270	94 275	94 280	94 285	94 290	94 295
877	94 300	94 305	94 310	94 315	94 320	94 325	94 330	94 335	94 340	94 345
878	94 349	94 354	94 359	94 364	94 369	94 374	94 379	94 384	94 389	94 394
879	94 399	94 404	94 409	94 414	94 419	94 424	94 429	94 433	94 438	94 443
880	94 448	94 453	94 458	94 463	94 468	94 473	94 478	94 483	94 488	94 493
881	94 498	94 503	94 507	94 512	94 517	94 522	94 527	94 532	94 537	94 542
882	94 547	94 552	94 557	94 562	94 567	94 571	94 576	94 581	94 586	94 591
883	94 596	94 601	94 606	94 611	94 616	94 621	94 626	94 630	94 635	94 640
884	94 645	94 650	94 655	94 660	94 665	94 670	94 675	94 680	94 685	94 689
885	94 694	94 699	94 704	94 709	94 714	94 719	94 724	94 729	94 734	94 738
886	94 743	94 748	94 753	94 758	94 763	94 768	94 773	94 778	94 783	94 787
887	94 792	94 797	94 802	94 807	94 812	94 817	94 822	94 827	94 832	94 836
888	94 841	94 846	94 851	94 856	94 861	94 866	94 871	94 876	94 880	94 885
889	94 890	94 895	94 900	94 905	94 910	94 915	94 919	94 924	94 929	94 934
890	94 939	94 944	94 949	94 954	94 959	94 963	94 968	94 973	94 978	94 983
891	94 988	94 993	94 998	95 002	95 007	95 012	95 017	95 022	95 027	95 032
892	95 036	95 041	95 046	95 051	95 056	95 061	95 066	95 071	95 075	95 080
893	95 085	95 090	95 095	95 100	95 105	95 109	95 114	95 119	95 124	95 129
894	95 134	95 139	95 143	95 148	95 153	95 158	95 163	95 168	95 173	95 177
895	95 182	95 187	95 192	95 197	95 202	95 207	95 211	95 216	95 221	95 226
896	95 231	95 236	95 240	95 245	95 250	95 255	95 260	95 265	95 270	95 274
897	95 279	95 284	95 289	95 294	95 299	95 303	95 308	95 313	95 318	95 323
898	95 328	95 332	95 337	95 342	95 347	95 352	95 357	95 361	95 366	95 371
899	95 376	95 381	95 386	95 390	95 395	95 400	95 405	95 410	95 415	95 419
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468

No.	0	1	2	3	4	5	6	7	8	9
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468
901	95 472	95 477	95 482	95 487	95 492	95 497	95 501	95 506	95 511	95 516
902	95 521	95 525	95 530	95 535	95 540	95 545	95 550	95 554	95 559	95 564
903	95 569	95 574	95 578	95 583	95 588	95 593	95 598	95 602	95 607	95 612
904	95 617	95 622	95 626	95 631	95 636	95 641	95 646	95 650	95 655	95 660
905	95 665	95 670	95 674	95 679	95 684	95 689	95 694	95 698	95 703	95 708
906	95 713	95 718	95 722	95 727	95 732	95 737	95 742	95 746	95 751	95 756
907	95 761	95 766	95 770	95 775	95 780	95 785	95 789	95 794	95 799	95 804
908	95 809	95 813	95 818	95 823	95 828	95 832	95 837	95 842	95 847	95 852
909	95 856	95 861	95 866	95 871	95 875	95 880	95 885	95 890	95 895	95 899
910	95 904	95 909	95 914	95 918	95 923	95 928	95 933	95 938	95 942	95 947
911	95 952	95 957	95 961	95 966	95 971	95 976	95 980	95 985	95 990	95 995
912	95 999	96 004	96 009	96 014	96 019	96 023	96 028	96 033	96 038	96 042
913	96 047	96 052	96 057	96 061	96 066	96 071	96 076	96 080	96 085	96 090
914	96 095	96 099	96 104	96 109	96 114	96 118	96 123	96 128	96 133	96 137
915	96 142	96 147	96 152	96 156	96 161	96 166	96 171	96 175	96 180	96 185
916	96 190	96 194	96 199	96 204	96 209	96 213	96 218	96 223	96 227	96 232
917	96 237	96 242	96 246	96 251	96 256	96 261	96 265	96 270	96 275	96 280
918	96 284	96 289	96 294	96 298	96 303	96 308	96 313	96 317	96 322	96 327
919	96 332	96 336	96 341	96 346	96 350	96 355	96 360	96 365	96 369	96 374
920	96 379	96 384	96 388	96 393	96 398	96 402	96 407	96 412	96 417	96 421
921	96 426	96 431	96 435	96 440	96 445	96 450	96 454	96 459	96 464	96 468
922	96 473	96 478	96 483	96 487	96 492	96 497	96 501	96 506	96 511	96 515
923	96 520	96 525	96 530	96 534	96 539	96 544	96 548	96 553	96 558	96 562
924	96 567	96 572	96 577	96 581	96 586	96 591	96 595	96 600	96 605	96 609
925	96 614	96 619	96 624	96 628	96 633	96 638	96 642	96 647	96 652	96 656
926	96 661	96 666	96 670	96 675	96 680	96 685	96 689	96 694	96 699	96 703
927	96 708	96 713	96 717	96 722	96 727	96 731	96 736	96 741	96 745	96 750
928	96 755	96 759	96 764	96 769	96 774	96 778	96 783	96 788	96 792	96 797
929	96 802	96 806	96 811	96 816	96 820	96 825	96 830	96 834	96 839	96 844
930	96 848	96 853	96 858	96 862	96 867	96 872	96 876	96 881	96 886	96 890
931	96 895	96 900	96 904	96 909	96 914	96 918	96 923	96 928	96 932	96 937
932	96 942	96 946	96 951	96 956	96 960	96 965	96 970	96 974	96 979	96 984
933	96 988	96 993	96 997	97 002	97 007	97 011	97 016	97 021	97 025	97 030
934	97 035	97 039	97 044	97 049	97 053	97 058	97 063	97 067	97 072	97 077
935	97 081	97 086	97 090	97 095	97 100	97 104	97 109	97 114	97 118	97 123
936	97 128	97 132	97 137	97 142	97 146	97 151	97 155	97 160	97 165	97 169
937	97 174	97 179	97 183	97 188	97 192	97 197	97 202	97 206	97 211	97 216
938	97 220	97 225	97 230	97 234	97 239	97 243	97 248	97 253	97 257	97 262
939	97 267	97 271	97 276	97 280	97 285	97 290	97 294	97 299	97 304	97 308
940	97 313	97 317	97 322	97 327	97 331	97 336	97 340	97 345	97 350	97 354
941	97 359	97 364	97 368	97 373	97 377	97 382	97 387	97 391	97 396	97 400
942	97 405	97 410	97 414	97 419	97 424	97 428	97 433	97 437	97 442	97 447
943	97 451	97 456	97 460	97 465	97 470	97 474	97 479	97 483	97 488	97 493
944	97 497	97 502	97 506	97 511	97 516	97 520	97 525	97 529	97 534	97 539
945	97 543	97 548	97 552	97 557	97 562	97 566	97 571	97 575	97 580	97 585
946	97 589	97 594	97 598	97 603	97 607	97 612	97 617	97 621	97 626	97 630
947	97 635	97 640	97 644	97 649	97 653	97 658	97 663	97 667	97 672	97 676
948	97 681	97 685	97 690	97 695	97 699	97 704	97 708	97 713	97 717	97 722
949	97 727	97 731	97 736	97 740	97 745	97 749	97 754	97 759	97 763	97 768
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813

No.	0	1	2	3	4	5	6	7	8	9
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813
951	97 818	97 823	97 827	97 832	97 836	97 841	97 845	97 850	97 855	97 859
952	97 864	97 868	97 873	97 877	97 882	97 886	97 891	97 896	97 900	97 905
953	97 909	97 914	97 918	97 923	97 928	97 932	97 937	97 941	97 946	97 950
954	97 955	97 959	97 964	97 968	97 973	97 978	97 982	97 987	97 991	97 996
955	98 000	98 005	98 009	98 014	98 019	98 023	98 028	98 032	98 037	98 041
956	98 046	98 050	98 055	98 059	98 064	98 068	98 073	98 078	98 082	98 087
957	98 091	98 096	98 100	98 105	98 109	98 114	98 118	98 123	98 127	98 132
958	98 137	98 141	98 146	98 150	98 155	98 159	98 164	98 168	98 173	98 177
959	98 182	98 186	98 191	98 195	98 200	98 204	98 209	98 214	98 218	98 223
960	98 227	98 232	98 236	98 241	98 245	98 250	98 254	98 259	98 263	98 268
961	98 272	98 277	98 281	98 286	98 290	98 295	98 299	98 304	98 308	98 313
962	98 318	98 322	98 327	98 331	98 336	98 340	98 345	98 349	98 354	98 358
963	98 363	98 367	98 372	98 376	98 381	98 385	98 390	98 394	98 399	98 403
964	98 408	98 412	98 417	98 421	98 426	98 430	98 435	98 439	98 444	98 448
965	98 453	98 457	98 462	98 466	98 471	98 475	98 480	98 484	98 489	98 493
966	98 498	98 502	98 507	98 511	98 516	98 520	98 525	98 529	98 534	98 538
967	98 543	98 547	98 552	98 556	98 561	98 565	98 570	98 574	98 579	98 583
968	98 588	98 592	98 597	98 601	98 605	98 610	98 614	98 619	98 623	98 628
969	98 632	98 637	98 641	98 646	98 650	98 655	98 659	98 664	98 668	98 673
970	98 677	98 682	98 686	98 691	98 695	98 700	98 704	98 709	98 713	98 717
971	98 722	98 726	98 731	98 735	98 740	98 744	98 749	98 753	98 758	98 762
972	98 767	98 771	98 776	98 780	98 784	98 789	98 793	98 798	98 802	98 807
973	98 811	98 816	98 820	98 825	98 829	98 834	98 838	98 843	98 847	98 851
974	98 856	98 860	98 865	98 869	98 874	98 878	98 883	98 887	98 892	98 896
975	98 900	98 905	98 909	98 914	98 918	98 923	98 927	98 932	98 936	98 941
976	98 945	98 949	98 954	98 958	98 963	98 967	98 972	98 976	98 981	98 985
977	98 989	98 994	98 998	99 003	99 007	99 012	99 016	99 021	99 025	99 029
978	99 034	99 038	99 043	99 047	99 052	99 056	99 061	99 065	99 069	99 074
979	99 078	99 083	99 087	99 092	99 096	99 100	99 105	99 109	99 114	99 118
980	99 123	99 127	99 131	99 136	99 140	99 145	99 149	99 154	99 158	99 162
981	99 167	99 171	99 176	99 180	99 185	99 189	99 193	99 198	99 202	99 207
982	99 211	99 216	99 220	99 224	99 229	99 233	99 238	99 242	99 247	99 251
983	99 255	99 260	99 264	99 269	99 273	99 277	99 282	99 286	99 291	99 295
984	99 300	99 304	99 308	99 313	99 317	99 322	99 326	99 330	99 335	99 339
985	99 344	99 348	99 352	99 357	99 361	99 366	99 370	99 374	99 379	99 383
986	99 388	99 392	99 396	99 401	99 405	99 410	99 414	99 419	99 423	99 427
987	99 432	99 436	99 441	99 445	99 449	99 454	99 458	99 463	99 467	99 471
988	99 476	99 480	99 484	99 489	99 493	99 498	99 502	99 506	99 511	99 515
989	99 520	99 524	99 528	99 533	99 537	99 542	99 546	99 550	99 555	99 559
990	99 564	99 568	99 572	99 577	99 581	99 585	99 590	99 594	99 599	99 603
991	99 607	99 612	99 616	99 621	99 625	99 629	99 634	99 638	99 642	99 647
992	99 651	99 656	99 660	99 664	99 669	99 673	99 677	99 682	99 686	99 691
993	99 695	99 699	99 704	99 708	99 712	99 717	99 721	99 726	99 730	99 734
994	99 739	99 743	99 747	99 752	99 756	99 760	99 765	99 769	99 774	99 778
995	99 782	99 787	99 791	99 795	99 800	99 804	99 808	99 813	99 817	99 822
996	99 826	99 830	99 835	99 839	99 843	99 848	99 852	99 856	99 861	99 865
997	99 870	99 874	99 878	99 883	99 887	99 891	99 896	99 900	99 904	99 909
998	99 913	99 917	99 922	99 926	99 930	99 935	99 939	99 944	99 948	99 952
999	99 957	99 961	99 965	99 970	99 974	99 978	99 983	99 987	99 991	99 996
1000	00 000	00 004	00 009	00 013	00 017	00 022	00 026	00 030	00 035	00 039

TABLE II

LOGARITHMS

OF THE

TRIGONOMETRIC FUNCTIONS

FOR

EACH MINUTE

′	0°				1°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	—	—	—	0.00 000	8.24 186	8.24 192	1.75 808	9.99 993	60
1	6 46 373	6.46 373	3.53 627	0.00 000	8.24 903	8.24 910	1.75 090	9.99 993	59
2	6.76 476	6.76 476	3.23 524	0.00 000	8.25 609	8.25 616	1.74 384	9.99 993	58
3	6.94 085	6.94 085	3.05 915	0.00 000	8.26 304	8.26 312	1.73 688	9.99 993	57
4	7.06 579	7.06 579	2.93 421	0.00 000	8.26 988	8.26 996	1.73 004	9.99 992	56
5	7.16 270	7.16 270	2.83 730	0.00 000	8.27 661	8.27 669	1.72 331	9.99 992	55
6	7.24 188	7.24 188	2.75 812	0.00 000	8.28 324	8.28 332	1.71 668	9.99 992	54
7	7.30 882	7.30 882	2.69 118	0.00 000	8.28 977	8.28 986	1.71 014	9.99 992	53
8	7.36 682	7.36 682	2.63 318	0.00 000	8.29 621	8.29 629	1.70 371	9.99 992	52
9	7.41 797	7.41 797	2.58 203	0.00 000	8.30 255	8.30 263	1.69 737	9.99 991	51
10	7.46 373	7.46 373	2.53 627	0.00 000	8.30 879	8.30 888	1.69 112	9.99 991	50
11	7.50 512	7.50 512	2.49 488	0.00 000	8.31 495	8.31 505	1.68 495	9.99 991	59
12	7.54 291	7.54 291	2.45 709	0.00 000	8.32 103	8.32 112	1.67 888	9.99 990	48
13	7.57 767	7.57 767	2.42 233	0.00 000	8.32 702	8.32 711	1.67 289	9.99 990	47
14	7.60 985	7.60 986	2.39 014	0.00 000	8.33 292	8.33 302	1.66 698	9.99 990	46
15	7.63 982	7.63 982	2.36 018	0.00 000	8.33 875	8.33 886	1.66 114	9.99 990	45
16	7.66 784	7.66 785	2.33 215	0.00 000	8.34 450	8.34 461	1.65 539	9.99 989	44
17	7.69 417	7.69 418	2.30 582	9.99 999	8.35 018	8.35 029	1.64 971	9.99 989	43
18	7.71 900	7.71 900	2.28 100	9.99 999	8.35 578	8.35 590	1.64 410	9.99 989	42
19	7.74 248	7.74 248	2.25 752	9.99 999	8.36 131	8.36 143	1.63 857	9.99 989	41
20	7.76 475	7.76 476	2.23 524	9.99 999	8.36 678	8.36 689	1.63 311	9.99 988	40
21	7.78 594	7.78 595	2.21 405	9.99 999	8.37 217	8.37 229	1.62 771	9.99 988	39
22	7.80 615	7.80 615	2.19 385	9.99 999	8.37 750	8.37 762	1.62 238	9.99 988	38
23	7.82 545	7.82 546	2.17 454	9.99 999	8.38 276	8.38 289	1.61 711	9.99 987	37
24	7.84 393	7.84 394	2.15 606	9.99 999	8.38 796	8.38 809	1.61 191	9.99 987	36
25	7.86 166	7.86 167	2.13 833	9.99 999	8.39 310	8.39 323	1.60 677	9.99 987	35
26	7.87 870	7.87 871	2.12 129	9.99 999	8.39 818	8.39 832	1.60 168	9.99 986	34
27	7.89 509	7.89 510	2.10 490	9.99 999	8.40 320	8.40 334	1.59 666	9.99 986	33
28	7.91 088	7.91 089	2.08 911	9.99 999	8.40 816	8.40 830	1.59 170	9.99 986	32
29	7.92 612	7.92 613	2.07 387	9.99 998	8.41 307	8.41 321	1.58 679	9.99 985	31
30	7.94 084	7.94 086	2.05 914	9.99 998	8.41 792	8.41 807	1.58 193	9.99 985	30
31	7.95 508	7.95 510	2.04 490	9.99 998	8.42 272	8.42 287	1.57 713	9.99 985	29
32	7.96 887	7.96 889	2.03 111	9.99 998	8.42 746	8.42 762	1.57 238	9.99 984	28
33	7.98 223	7.98 225	2.01 775	9.99 998	8.43 216	8.43 232	1.56 768	9.99 984	27
34	7.99 520	7.99 522	2.00 478	9.99 998	8.43 680	8.43 696	1.56 304	9.99 984	26
35	8.00 779	8.00 781	1.99 219	9.99 998	8.44 139	8.44 156	1.55 844	9.99 983	25
36	8.02 002	8.02 004	1.97 996	9.99 998	8.44 594	8.44 611	1.55 389	9.99 983	24
37	8.03 192	8.03 194	1.96 806	9.99 997	8.45 044	8.45 061	1.54 939	9.99 983	23
38	8.04 350	8.04 353	1.95 647	9.99 997	8.45 489	8.45 507	1.54 493	9.99 982	22
39	8.05 478	8.05 481	1.94 519	9.99 997	8.45 930	8.45 948	1.54 052	9.99 982	21
40	8.06 578	8.06 581	1.93 419	9.99 997	8.46 366	8.46 385	1.53 615	9.99 982	20
41	8.07 650	8.07 653	1.92 347	9.99 997	8.46 799	8.46 817	1.53 183	9.99 981	19
42	8.08 696	8.08 700	1.91 300	9.99 997	8.47 226	8.47 245	1.52 755	9.99 981	18
43	8.09 718	8.09 722	1.90 278	9.99 997	8.47 650	8.47 669	1.52 331	9.99 981	17
44	8.10 717	8.10 720	1.89 280	9.99 996	8.48 069	8.48 089	1.51 911	9.99 980	16
45	8.11 693	8.11 696	1.88 304	9.99 996	8.48 485	8.48 505	1.51 495	9.99 980	15
46	8.12 647	8.12 651	1.87 349	9.99 996	8.48 896	8.48 917	1.51 083	9.99 979	14
47	8.13 581	8.13 585	1.86 415	9.99 996	8.49 304	8.49 325	1.50 675	9.99 979	13
48	8.14 495	8.14 500	1.85 500	9.99 996	8.49 708	8.49 729	1.50 271	9.99 979	12
49	8.15 391	8.15 395	1.84 605	9.99 996	8.50 108	8.50 130	1.49 870	9.99 978	11
50	8.16 268	8.16 273	1.83 727	9.99 995	8.50 504	8.50 527	1.49 473	9.99 978	10
51	8.17 128	8.17 133	1.82 867	9.99 995	8.50 897	8.50 920	1.49 080	9.99 977	9
52	8.17 971	8.17 976	1.82 024	9.99 995	8.51 287	8.51 310	1.48 690	9.99 977	8
53	8.18 798	8.18 804	1.81 196	9.99 995	8.51 673	8.51 696	1.48 304	9.99 977	7
54	8.19 610	8.19 616	1.80 384	9.99 995	8.52 055	8.52 079	1.47 921	9.99 976	6
55	8 20 407	8.20 413	1.79 587	9.99 994	8 52 434	8.52 459	1.47 541	9.99 976	5
56	8.21 189	8.21 195	1.78 805	9.99 994	8.52 810	8.52 835	1.47 165	9.99 975	4
57	8.21 958	8.21 964	1.78 036	9.99 994	8.53 183	8.53 208	1.46 792	9.99 975	3
58	8.22 713	8.22 720	1.77 280	9.99 994	8.53 552	8.53 578	1.46 422	9.99 974	2
59	8.23 456	8.23 462	1.76 538	9.99 994	8.53 919	8.53 945	1.46 055	9.99 974	1
60	8.24 186	8.24 192	1.75 808	9.99 993	8.54 282	8.54 308	1.45 692	9.99 974	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	89°				88°				′

′	1°				2°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	8.24 186	8.24 192	1.75 808	9.99 993	8.54 282	8.54 308	1.45 692	9.99 974	60
1	8.24 903	8.24 910	1.75 090	9.99 993	8.54 642	8.54 669	1.45 331	9.99 973	59
2	8.25 609	8.25 616	1.74 384	9.99 993	8.54 999	8.55 027	1.44 973	9.99 973	58
3	8.26 304	8.26 312	1.73 688	9.99 993	8.55 354	8.55 382	1.44 618	9.99 972	57
4	8.26 988	8.26 996	1.73 004	9.99 992	8.55 705	8.55 734	1.44 266	9.99 972	56
5	8.27 661	8.27 669	1.72 331	9.99 992	8.56 054	8.56 083	1.43 917	9.99 971	55
6	8.28 324	8.28 332	1.71 668	9.99 992	8.56 400	8.56 429	1.43 571	9.99 971	54
7	8.28 977	8.28 986	1.71 014	9.99 992	8.56 743	8.56 773	1.43 227	9.99 970	53
8	8.29 621	8.29 629	1.70 371	9.99 992	8.57 084	8.57 114	1.42 886	9.99 970	52
9	8.30 255	8.30 263	1.69 737	9.99 991	8.57 421	8.57 452	1.42 548	9.99 969	51
10	8.30 879	8.30 888	1.69 112	9.99 991	8.57 757	8.57 788	1.42 212	9.99 969	50
11	8.31 495	8.31 505	1.68 495	9.99 991	8.58 089	8.58 121	1.41 879	9.99 968	49
12	8.32 103	8.32 112	1.67 888	9.99 990	8.58 419	8.58 451	1.41 549	9.99 968	48
13	8.32 702	8.32 711	1.67 289	9.99 990	8.58 747	8.58 779	1.41 221	9.99 967	47
14	8.33 292	8.33 302	1.66 698	9.99 990	8.59 072	8.59 105	1.40 895	9.99 967	46
15	8.33 875	8.33 886	1.66 114	9.99 990	8.59 395	8.59 428	1.40 572	9.99 967	45
16	8.34 450	8.34 461	1.65 539	9.99 989	8.59 715	8.59 749	1.40 251	9.99 966	44
17	8.35 018	8.35 029	1.64 971	9.99 989	8.60 033	8.60 068	1.39 932	9.99 966	43
18	8.35 578	8 35 590	1.64 410	9.99 989	8.60 349	8.60 384	1.39 616	9.99 965	42
19	8.36 131	8.36 143	1.63 857	9.99 989	8.60 662	8.60 698	1.39 302	9.99 964	41
20	8.36 678	8.36 689	1.63 311	9.99 988	8.60 973	8.61 009	1.38 991	9.99 964	40
21	8.37 217	8.37 229	1.62 771	9.99 988	8.61 282	8.61 319	1.38 681	9.99 963	39
22	8.37 750	8.37 762	1.62 238	9.99 988	8.61 589	8 61.626	1.38 374	9.99 963	38
23	8.38 276	8 38 289	1.61 711	9.99 987	8.61 894	8.61 931	1.38 069	9 99 962	37
24	8.38 796	8.38 809	1.61 191	9.99 987	8.62 196	8.62 234	1.37 766	9.99 962	36
25	8.39 310	8.39 323	1.60 677	9.99 987	8.62 497	8.62 535	1.37 465	9.99 961	35
26	8.39 818	8.39 832	1.60 168	9.99 986	8.62 795	8.62 834	1.37 166	9.99 961	34
27	8.40 320	8.40 334	1.59 666	9.99 986	8.63 091	8.63 131	1.36 869	9.99 960	33
28	8.40 816	8.40 830	1.59 170	9.99 986	8.63 385	8.63 426	1.36 574	9.99 960	32
29	8.41 307	8.41 321	1.58 679	9.99 985	8.63 678	8.63 718	1.36 282	9.99 959	31
30	8.41 792	8.41 807	1.58 193	9.99 985	8.63 968	8.64 009	1.35 991	9.99 959	30
31	8.42 272	8.42 287	1.57 713	9.99 985	8.64 256	8.64 298	1.35 702	9 99 958	29
32	8.42 746	8.42 762	1.57 238	9.99 984	8.64 543	8.64 585	1.35 415	9.99 958	28
33	8.43 216	8.43 232	1.56 768	9.99 984	8.64 827	8.64 870	1.35 130	9.99 957	27
34	8.43 680	8.43 696	1.56 304	9.99 984	8.65 110	8.65 154	1.3 4846	9.99 956	26
35	8.44 139	8.44 156	1.55 844	9.99 983	8.65 391	8.65 435	1.34 565	9.99 956	25
36	8.44 594	8.44 611	1.55 389	9.99 983	8.65 670	8.65 715	1.34 285	9.99 955	24
37	8.45 044	8.45 061	1.54 939	9.99 983	8.65 947	8.65 993	1.34 007	9.99 955	23
38	8.45 489	8.45 507	1.54 493	9.99 982	8.66 223	8.66 269	1.33 731	9.99 954	22
39	8.45 930	8.45 948	1.54 052	9.99 982	8.66 497	8.66 543	1.33 457	9.99 954	21
40	8.46 366	8.46 385	1.53 615	9.99 982	8.66 769	8.66 816	1.33 184	9.99 953	20
41	8.46 799	8.46 817	1.53 183	9.99 981	8.67 039	8.67 087	1.32 913	9.99 952	19
42	8.47 226	8.47 245	1.52 755	9.99 981	8.67 308	8.67 356	1.32 644	9.99 952	18
43	8.47 650	8.47 669	1.52 331	9.99 981	8.67 575	8.67 624	1.32 376	9.99 951	17
44	8.48 069	8.48 089	1.51 911	9.99 980	8.67 841	8.67 890	1.32 110	9.99 951	16
45	8.48 485	8 48 505	1.51 495	9.99 980	8.68 104	8.68 154	1.31 846	9.99 950	15
46	8.48 896	8.48 917	1.51 083	9.99 979	8.68 367	8.68 417	1.31 583	9.99 949	14
47	8.49 304	8.49 325	1.50 675	9.99 979	8.68 627	8.68 678	1.31 322	9.99 949	13
48	8.49 708	8.49 729	1.50 271	9.99 979	8.68 886	8.68 938	1.31 062	9.99 948	12
49	8.50 108	8.50 130	1.49 870	9.99 978	8.69 144	8.69 196	1.30 804	9.99 948	11
50	8.50 504	8.50 527	1.49 473	9.99 978	8.69 400	8.69 453	1.30 547	9.99 947	10
51	8.50 897	8.50 920	1.49 080	9.99 977	8.69 654	8.69 708	1.30 292	9.99 946	9
52	8.51 287	8.51 310	1.48 690	9.99 977	8.69 907	8.69 962	1.30 038	9.99 946	8
53	8.51 673	8.51 696	1.48 304	9.99 977	8.70 159	8.70 214	1.29 786	9.99 945	7
54	8.52 055	8.52 079	1.47 921	9.99 976	8.70 409	8.70 465	1.29 535	9.99 944	6
55	8.52 434	8.52 459	1.47 541	9.99 976	8.70 658	8.70 714	1.29 286	9.99 944	5
56	8.52 810	8.52 835	1.47 165	9.99 975	8.70 905	8.70 962	1.29 038	9.99 943	4
57	8.53 183	8.53 208	1.46 792	9.99 975	8.71 151	8.71 208	1.28 792	9.99 942	3
58	8.53 552	8.53 578	1.46 422	9.99 974	8.71 395	8.71 453	1.28 547	9.99 942	2
59	8.53 919	8.53 945	1.46 055	9.99 974	8.71 638	8.71 697	1.28 303	9.99 941	1
60	8.54 282	8.54 308	1.45 692	9.99 974	8.71 880	8.71 940	1.28 060	9.99 940	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	88°				87°				′

′	3°				4°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	8.71 880	8.71 940	1.28 060	9.99 940	8.84 358	8.84 464	1.15 536	9.99 894	**60**
1	8.72 120	8.72 181	1.27 819	9.99 940	8.84 539	8.84 646	1.15 354	9.99 893	59
2	8.72 359	8.72 420	1.27 580	9.99 939	8.84 718	8.84 826	1.15 174	9.99 892	58
3	8.72 597	8.72 659	1.27 341	9.99 938	8.84 897	8.85 006	1.14 994	9.99 891	57
4	8.72 834	8.72 896	1.27 104	9.99 938	8.85 075	8.85 185	1.14 815	9.99 891	56
5	8.73 069	8.73 132	1.26 868	9.99 937	8.85 252	8.85 363	1.14 637	9.99 890	55
6	8.73 303	8.73 366	1.26 634	9.99 936	8.85 429	8.85 540	1.14 460	9.99 889	54
7	8.73 535	8.73 600	1.26 400	9.99 936	8.85 605	8.85 717	1.14 283	9.99 888	53
8	8.73 767	8.73 832	1.26 168	9.99 935	8.85 780	8.85 893	1.14 107	9.99 887	52
9	8.73 997	8.74 063	1.25 937	9.99 934	8.85 955	8.86 069	1.13 931	9.99 886	51
10	8.74 226	8.74 292	1.25 708	9.99 934	8.86 128	8.86 243	1.13 757	9.99 885	**50**
11	8.74 454	8.74 521	1.25 479	9.99 933	8.86 301	8.86 417	1.13 583	9.99 884	49
12	8.74 680	8.74 748	1.25 252	9.99 932	8.86 474	8.86 591	1.13 409	9.99 883	48
13	8.74 906	8.74 974	1.25 026	9.99 932	8.86 645	8.86 763	1.13 237	9.99 882	47
14	8.75 130	8.75 199	1.24 801	9.99 931	8.86 816	8.86 935	1.13 065	9.99 881	46
15	8.75 353	8.75 423	1.24 577	9.99 930	8.86 987	8.87 106	1.12 894	9.99 880	45
16	8.75 575	8.75 645	1.24 355	9.99 929	8.87 156	8.87 277	1.12 723	9.99 879	44
17	8.75 795	8.75 867	1.24 133	9.99 929	8.87 325	8.87 447	1.12 553	9.99 879	43
18	8.76 015	8.76 087	1.23 913	9.99 928	8.87 494	8.87 616	1.12 384	9.99 878	42
19	8.76 234	8.76 306	1.23 694	9.99 927	8.87 661	8.87 785	1.12 215	9.99 877	41
20	8.76 451	8.76 525	1.23 475	9.99 926	8.87 829	8.87 953	1.12 047	9.99 876	**40**
21	8.76 667	8.76 742	1.23 258	9.99 926	8.87 995	8.88 120	1.11 880	9.99 875	39
22	8.76 883	8.76 958	1.23 042	9.99 925	8.88 161	8.88 287	1.11 713	9.99 874	38
23	8.77 097	8.77 173	1.22 827	9.99 924	8.88 326	8.88 453	1.11 547	9.99 873	37
24	8.77 310	8.77 387	1.22 613	9.99 923	8.88 490	8.88 618	1.11 382	9.99 872	36
25	8.77 522	8.77 600	1.22 400	9.99 923	8.88 654	8.88 783	1.11 217	9.99 871	35
26	8.77 733	8.77 811	1.22 189	9.99 922	8.88 817	8.88 948	1.11 052	9.99 870	34
27	8.77 943	8.78 022	1.21 978	9.99 921	8.88 980	8.89 111	1.10 889	9.99 869	33
28	8.78 152	8.78 232	1.21 768	9.99 920	8.89 142	8.89 274	1.10 726	9.99 868	32
29	8.78 360	8.78 441	1.21 559	9.99 920	8.89 304	8.89 437	1.10 563	9.99 867	31
30	8.78 568	8.78 649	1.21 351	9.99 919	8.89 464	8.89 598	1.10 402	9.99 866	**30**
31	8.78 774	8.78 855	1.21 145	9.99 918	8.89 625	8.89 760	1.10 240	9.99 865	29
32	8.78 979	8.79 061	1.20 939	9.99 917	8.89 784	8.89 920	1.10 080	9.99 864	28
33	8.79 183	8.79 266	1.20 734	9.99 917	8.89 943	8.90 080	1.09 920	9.99 863	27
34	8.79 386	8.79 470	1.20 530	9.99 916	8.90 102	8.90 240	1.09 760	9.99 862	26
35	8.79 588	8.79 673	1.20 327	9.99 915	8.90 260	8.90 399	1.09 601	9.99 861	25
36	8.79 789	8.79 875	1.20 125	9.99 914	8.90 417	8.90 557	1.09 443	9.99 860	24
37	8.79 990	8.80 076	1.19 924	9.99 913	8.90 574	8.90 715	1.09 285	9.99 859	23
38	8.80 189	8.80 277	1.19 723	9.99 913	8.90 730	8.90 872	1.09 128	9.99 858	22
39	8.80 388	8.80 476	1.19 524	9.99 912	8.90 885	8.91 029	1.08 971	9.99 857	21
40	8.80 585	8.80 674	1.19 326	9.99 911	8.91 040	8.91 185	1.08 815	9.99 856	**20**
41	8.80 782	8.80 872	1.19 128	9.99 910	8.91 195	8.91 340	1.08 660	9.99 855	19
42	8.80 978	8.81 068	1.18 932	9.99 909	8.91 349	8.91 495	1.08 505	9.99 854	18
43	8.81 173	8.81 264	1.18 736	9.99 909	8.91 502	8.91 650	1.08 350	9.99 853	17
44	8.81 367	8.81 459	1.18 541	9.99 908	8.91 655	8.91 803	1.08 197	9.99 852	16
45	8.81 560	8.81 653	1.18 347	9.99 907	8.91 807	8.91 957	1.08 043	9.99 851	15
46	8.81 752	8.81 846	1.18 154	9.99 906	8.91 959	8.92 110	1.07 890	9.99 850	14
47	8.81 944	8.82 038	1.17 962	9.99 905	8.92 110	8.92 262	1.07 738	9.99 848	13
48	8.82 134	8.82 230	1.17 770	9.99 904	8.92 261	8.92 414	1.07 586	9.99 847	12
49	8.82 324	8.82 420	1.17 580	9.99 904	8.92 411	8.92 565	1.07 435	9.99 846	11
50	8.82 513	8.82 610	1.17 390	9.99 903	8.92 561	8.92 716	1.07 284	9.99 845	**10**
51	8.82 701	8.82 799	1.17 201	9.99 902	8.92 710	8.92 866	1.07 134	9.99 844	9
52	8.82 888	8.82 987	1.17 013	9.99 901	8.92 859	8.93 016	1.06 984	9.99 843	8
53	8.83 075	8.83 175	1.16 825	9.99 900	8.93 007	8.93 165	1.06 835	9.99 842	7
54	8.83 261	8.83 361	1.16 639	9.99 899	8.93 154	8.93 313	1.06 687	9.99 841	6
55	8.83 446	8.83 547	1.16 453	9.99 898	8.93 301	8.93 462	1.06 538	9.99 840	5
56	8.83 630	8.83 732	1.16 268	9.99 898	8.93 448	8.93 609	1.06 391	9.99 839	4
57	8.83 813	8.83 916	1.16 084	9.99 897	8.93 594	8.93 756	1.06 244	9.99 838	3
58	8.83 996	8.84 100	1.15 900	9.99 896	8.93 740	8.93 903	1.06 097	9.99 837	2
59	8.84 177	8.84 282	1.15 718	9.99 895	8.93 885	8.94 049	1.05 951	9.99 836	1
60	8.84 358	8.84 464	1.15 536	9.99 894	8.94 030	8.94 195	1.05 805	9.99 834	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	86°				85°				′

′	5°				6°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	8.94 030	8.94 195	1.05 805	9.99 834	9.01 923	9.02 162	0.97 838	9.99 761	**60**
1	8.94 174	8.94 340	1.05 660	9.99 833	9.02 043	9.02 283	0.97 717	9.99 760	59
2	8.94 317	8.94 485	1.05 515	9.99 832	9.02 163	9.02 404	0 97 596	9.99 759	58
3	8 94 461	8.94 630	1.05 370	9.99 831	9.02 283	9.02 525	0.97 475	9.99 757	57
4	8.94 603	8.94 773	1.05 227	9.99 830	9.02 402	9.02 645	0.97 355	9.99 756	56
5	8.94 746	8.94 917	1.05 083	9.99 829	9.02 520	9.02 766	0.97 234	9.99 755	55
6	8.94 887	8.95 060	1.04 940	9.99 828	9.02 639	9.02 885	0.97 115	9.99 753	54
7	8.95 029	8.95 202	1.04 798	9.99 827	9.02 757	9.03 005	0.96 995	9.99 752	53
8	8.95 170	8.95 344	1.04 656	9.99 825	9.02 874	9.03 124	0.96 876	9.99 751	52
9	8.95 310	8.95 486	1 04 514	9.99 824	9.02 992	9.03 242	0.96 758	9.99 749	51
10	8.95 450	8.95 627	1.04 373	9.99 823	9.03 109	9.03 361	0.96 639	9.99 748	**50**
11	8.95 589	8.95 767	1.04 233	9.99 822	9.03 226	9.03 479	0.96 521	9.99 747	49
12	8.95 728	8.95 908	1.04 092	9.99 821	9.03 342	9.03 597	0.96 403	9.99 745	48
13	8.95 867	8.96 047	1.03 953	9.99 820	9.03 458	9.03 714	0.96 286	9.99 744	47
14	8.96 005	8.96 187	1.03 813	9.99 819	9.03 574	9.03 832	0.96 168	9.99 742	46
15	8 96 143	8.96 325	1.03 675	9.99 817	9.03 690	9.03 948	0.96 052	9.99 741	45
16	8.96 280	8.96 464	1.03 536	9.99 816	9.03 805	9.04 065	0.95 935	9.99 740	44
17	8.96 417	8.96 602	1.03 398	9.99 815	9.03 920	9.04 181	0.95 819	9.99 738	43
18	8.96 553	8.96 739	1.03 261	9.99 814	9.04 034	9.04 297	0.95 703	9.99 737	42
19	8.96 689	8.96 877	1.03 123	9.99 813	9.04 149	9.04 413	0.95 587	9.99 736	41
20	8.96 825	8.97 013	1.02 987	9.99 812	9.04 262	9.04 528	0.95 472	9.99 734	**40**
21	8.96 960	8.97 150	1.02 850	9.99 810	9.04 376	9.04 643	0.95 357	9.99 733	39
22	8.97 095	8.97 285	1.02 715	9.99 809	9.04 490	9.04 758	0.95 242	9.99 731	38
23	8.97 229	8.97 421	1.02 579	9.99 808	9.04 603	9.04 873	0.95 127	9.99 730	37
24	8.97 363	8.97 556	1.02 444	9.99 807	9.04 715	9.04 987	0.95 013	9.99 728	36
25	8.97 496	8.97 691	1.02 309	9.99 806	9.04 828	9.05 101	0.94 899	9.99 727	35
26	8.97 629	8.97 825	1.02 175	9.99 804	9.04 940	9.05 214	0.94 786	9.99 726	34
27	8.97 762	8.97 959	1.02 041	9.99 803	9.05 052	9.05 328	0.94 672	9.99 724	33
28	8.97 894	8.98 092	1.01 908	9.99 802	9.05 164	9.05 441	0.94 559	9.99 723	32
29	8.98 026	8.98 225	1.01 775	9.99 801	9.05 275	9.05 553	0.94 447	9.99 721	31
30	8.98 157	8.98 358	1.01 642	9.99 800	9.05 386	9.05 666	0.94 334	9.99 720	**30**
31	8.98 288	8.98 490	1.01 510	9.99 798	9.05 497	9.05 778	0.94 222	9.99 718	29
32	8.98 419	8.98 622	1.01 378	9.99 797	9.05 607	9.05 890	0.94 110	9.99 717	28
33	8.98 549	8.98 753	1.01 247	9.99 796	9.05 717	9.06 002	0.93 998	9.99 716	27
34	8.98 679	8.98 884	1.01 116	9.99 795	9.05 827	9.06 113	0.93 887	9.99 714	26
35	8.98 808	8.99 015	1.00 985	9.99 793	9.05 937	9.06 224	0.93 776	9.99 713	25
36	8.98 937	8.99 145	1.00 855	9.99 792	9.06 046	9.06 335	0.93 665	9.99 711	24
37	8.99 066	8.99 275	1.00 725	9.99 791	9.06 155	9.06 445	0.93 555	9.99 710	23
38	8.99 194	8.99 405	1.00 595	9.99 790	9.06 264	9.06 556	0.93 444	9.99 708	22
39	8.99 322	8.99 534	1.00 466	9.99 788	9.06 372	9.06 666	0.93 334	9.99 707	21
40	8.99 450	8.99 662	1.00 338	9.99 787	9.06 481	9.06 775	0.93 225	9.99 705	**20**
41	8.99 577	8.99 791	1.00 209	9.99 786	9.06 589	9.06 885	0.93 115	9.99 704	19
42	8.99 704	8.99 919	1.00 081	9.99 785	9.06 696	9.06 994	0.93 006	9.99 702	18
43	8.99 830	9.00 046	0.99 954	9.99 783	9.06 804	9.07 103	0.92 897	9.99 701	17
44	8.99 956	9.00 174	0.99 826	9.99 782	9.06 911	9.07 211	0.92 789	9.99 699	16
45	9.00 082	9.00 301	0.99 699	9.99 781	9.07 018	9.07 320	0.92 680	9.99 698	15
46	9.00 207	9.00 427	0.99 573	9.99 780	9.07 124	9.07 428	0.92 572	9.99 696	14
47	9.00 332	9.00 553	0.99 447	9.99 778	9.07 231	9.07 536	0.92 464	9.99 695	13
48	9.00 456	9.00 679	0.99 321	9.99 777	9.07 337	9.07 643	0.92 357	9.99 693	12
49	9.00 581	9.00 805	0.99 195	9.99 776	9.07 442	9.07 751	0.92 249	9.99 692	11
50	9.00 704	9.00 930	0.99 070	9.99 775	9.07 548	9.07 858	0.92 142	9.99 690	**10**
51	9.00 828	9.01 055	0.98 945	9.99 773	9.07 653	9.07 964	0.92 036	9.99 689	9
52	9.00 951	9.01 179	0.98 821	9.99 772	9.07 758	9.08 071	0.91 929	9.99 687	8
53	9.01 074	9.01 303	0.98 697	9.99 771	9.07 863	9.08 177	0.91 823	9.99 686	7
54	9.01 196	9.01 427	0.98 573	9.99 769	9.07 968	9.08 283	0.91 717	9.99 684	6
55	9.01 318	9.01 550	0.98 450	9.99 768	9.08 072	9.08 389	0.91 611	9.99 683	5
56	9.01 440	9.01 673	9.98 327	9.99 767	9.08 176	9.08 495	0.91 505	9.99 681	4
57	9.01 561	9.01 796	0.98 204	9.99 765	9.08 280	9.08 600	0.91 400	9.99 680	3
58	9.01 682	9.01 918	0.98 082	9.99 764	9.08 383	9.08 705	0.91 295	9.99 678	2
59	9.01 803	9.02 040	0.97 960	9.99 763	9.08 486	9.08 810	0.91 190	9.99 677	1
60	9.01 923	9.02 162	0.97 838	9.99 761	9.08 589	9.08 914	0.91 086	9.99 675	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	84°				83°				′

′	7°				8°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.08 589	9.08 914	0 91 086	9.99 675	9.14 356	9.14 780	0.85 220	9.99 575	**60**
1	9.08 692	9.09 019	0 90 981	9.99 674	9.14 445	9.14 872	0.85 128	9.99 574	59
2	9.08 795	9.09 123	0.90 877	9.99 672	9.14 535	9.14 963	0.85 037	9.99 572	58
3	9.08 897	9.09 227	0.90 773	9.99 670	9.14 624	9.15 054	0.84 946	9.99 570	57
4	9.09 999	9.09 330	0.90 670	9.99 669	9.14 714	9.15 145	0.84 855	9.99 568	56
5	9.09 101	9.09 434	0.90 566	9.99 667	9.14 803	9.15 236	0.84 764	9.99 566	55
6	9.09 202	9.09 537	0.90 463	9.99 666	9.14 891	9.15 327	0.84 673	9.99 565	54
7	9.09 304	9.09 640	0.90 360	9.99 664	9.14 980	9.15 417	0.84 583	9.99 563	53
8	9.09 405	9.09 742	0.90 258	9.99 663	9.15 069	9.15 508	0.84 492	9.99 561	52
9	9.09 506	9.09 845	0.90 155	9.99 661	9.15 157	9.15 598	0.84 402	9.99 559	51
10	9.09 606	9.09 947	0.90 053	9.99 659	9.15 245	9.15 688	0.84 312	9.99 557	**50**
11	9.09 707	9.10 049	0.89 951	9.99 658	9.15 333	9.15 777	0.84 223	9.99 556	49
12	9.09 807	9.10 150	0.89 850	9.99 656	9.15 421	9.15 867	0.84 133	9.99 554	48
13	9.09 907	9.10 252	0.89 748	9.99 655	9.15 508	9.15 956	0.84 044	9.99 552	47
14	9.10 006	9.10 353	0.89 647	9.99 653	9.15 596	9.16 046	0.83 954	9.99 550	46
15	9.10 106	9.10 454	0.89 546	9.99 651	9.15 683	9.16 135	0.83 865	9.99 548	45
16	9.10 205	9.10 555	0.89 445	9.99 650	9.15 770	9.16 224	0.83 776	9.99 546	44
17	9.10 304	9.10 656	0.89 344	9.99 648	9.15 857	9.16 312	0.83 688	9.99 545	43
18	9.10 402	9.10 756	0.89 244	9.99 647	9.15 944	9.16 401	0.83 599	9.99 543	42
19	9.10 501	9.10 856	0.89 144	9.99 645	9.16 030	9.16 489	0.83 511	9.99 541	41
20	9.10 599	9.10 956	0.89 044	9.99 643	9.16 116	9.16 577	0.83 423	9.99 539	**40**
21	9.10 697	9.11 056	0.88 944	9.99 642	9.16 203	9.16 665	0.83 335	9.99 537	39
22	9.10 795	9.11 155	0.88 845	9.99 640	9.16 289	9.16 753	0.83 247	9.99 535	38
23	9.10 893	9.11 254	0.88 746	9.99 638	9.16 374	9.16 841	0.83 159	9.99 533	37
24	9.10 990	9.11 353	0.88 647	9.99 637	9.16 460	9.16 928	0.83 072	9.99 532	36
25	9.11 087	9.11 452	0.88 548	9.99 635	9.16 545	9.17 016	0.82 984	9.99 530	35
26	9.11 184	9.11 551	0.88 449	9.99 633	9.16 631	9.17 103	0.82 897	9.99 528	34
27	9.11 281	9.11 649	0.88 351	9.99 632	9.16 716	9.17 190	0.82 810	9.99 526	33
28	9.11 377	9.11 747	0.88 253	9.99 630	9.16 801	9.17 277	0.82 723	9.99 524	32
29	9.11 474	9.11 845	0.88 155	9.99 629	9.16 886	9.17 363	0.82 637	9.99 522	31
30	9.11 570	9.11 943	0.88 057	9.99 627	9.16 970	9.17 450	0.82 550	9.99 520	**30**
31	9.11 666	9.12 040	0.87 960	9.99 625	9.17 055	9.17 536	0.82 464	9.99 518	29
32	9.11 761	9.12 138	0.87 862	9.99 624	9.17 139	9.17 622	0.82 378	9.99 517	28
33	9.11 857	9.12 235	0.87 765	9.99 622	9.17 223	9.17 708	9.82 292	9.99 515	27
34	9.11 952	9.12 332	0.87 668	9.99 620	9.17 307	9.17 794	0.82 206	9.99 513	26
35	9.12 047	9.12 428	0.87 572	9.99 618	9.17 391	9.17 880	0.82 120	9 99 511	25
36	9.12 142	9.12 525	0.87 475	9.99 617	9.17 474	9.17 965	0.82 035	9.99 509	24
37	9.12 236	9.12 621	0.87 379	9.99 615	9.17 558	9.18 051	0.81 949	9.99 507	23
38	9.12 331	9.12 717	0.87 283	9.99 613	9.17 641	9.18 136	0.81 864	9.99 505	22
39	9.12 425	9.12 813	0.87 187	9.99 612	9.17 724	9.18 221	0.81 779	9.99 503	21
40	9.12 519	9.12 909	0.87 091	9.99 610	9.17 807	9.18 306	0.81 694	9.99 501	**20**
41	9.12 612	9.13 004	0.86 996	9.99 608	9.17 890	9.18 391	0.81 609	9.99 499	19
42	9.12 706	9.13 099	0.86 901	9.99 607	9.17 973	9.18 475	0.81 525	9.99 497	18
43	9.12 799	9.13 194	0.86 806	9.99 605	9.18 055	9.18 560	0.81 440	9 99 495	17
44	9.12 892	9.13 289	0.86 711	9.99 603	9.18 137	9.18 644	0.81 356	9.99 494	16
45	9.12 985	9.13 384	0.86 616	9.99 601	9.18 220	9.18 728	0.81 272	9.99 492	**15**
46	9.13 078	9.13 478	0.86 522	9.99 600	9.18 302	9.18 812	0.81 188	9.99 490	14
47	9.13 171	9.13 573	0.86 427	9.99 598	9.18 383	9.18 896	0.81 104	9.99 488	13
48	9.13 263	9.13 667	0.86 333	9.99 596	9.18 465	9.18 979	0.81 021	9.99 486	12
49	9.13 355	9.13 761	0.86 239	9.99 595	9.18 547	9.19 063	0.80 937	9.99 484	11
50	9.13 447	9.13 854	0.86 146	9.99 593	9.18 628	9.19 146	0.80 854	9.99 482	**10**
51	9.13 539	9.13 948	0 86 052	9.99 591	9.18 709	9.19 229	0.80 771	9.99 480	9
52	9.13 630	9.14 041	0.85 959	9.99 589	9.18 790	9.19 312	0.80 688	9.99 478	8
53	9.13 722	9.14 134	0.85 866	9.99 588	9.18 871	9.19 395	0.80 605	9.99 476	7
54	9.13 813	9.14 227	0.85 773	9.99 586	9.18 952	9.19 478	0.80 522	9.99 474	6
55	9.13 904	9.14 320	0.85 680	9.99 584	9.19 033	9.19 561	0.80 439	9.99 472	5
56	9.13 994	9.14 412	0.85 588	9.99 582	9.19 113	9.19 643	0.80 357	9.99 470	4
57	9.14 085	9.14 504	0.85 496	9.99 581	9.19 193	9.19 725	0.80 275	9.99 468	3
58	9.14 175	9.14 597	0.85 403	9.99 579	9.19 273	9.19 807	0.80 193	9.99 466	2
59	9.14 266	9.14 688	0.85 312	9.99 577	9.19 353	9.19 889	0.80 111	9.99 464	1
60	9.14 356	9.14 780	0.85 220	9.99 575	9.19 433	9 19 971	0.80 029	9.99 462	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	82°				81°				′

′	9°				10°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.19 433	9.19 971	0.80 209	9.99 462	9.23 967	9.24 632	0.75 368	9.99 335	**60**
1	9.19 513	9.20 053	0.79 947	9.99 460	9.24 039	9.24 706	0.75 294	9.99 333	59
2	9.19 592	9.20 134	0.79 866	9.99 458	9.24 110	9.24 779	0.75 221	9.99 331	58
3	9.19 672	9.20 216	0.79 784	9.99 456	9.24 181	9.24 853	0.75 147	9.99 328	57
4	9.19 751	9.20 297	0.79 703	9.99 454	9.24 253	9.24 926	0.75 074	9.99 326	56
5	9.19 830	9.20 378	0.79 622	9.99 452	9.24 324	9.25 000	0.75 000	9.99 324	55
6	9.19 909	9.20 495	0.79 541	9.99 450	9.24 395	9.25 073	0.74 927	9.99 322	54
7	9.19 988	9.20 540	0.79 460	9.99 448	9.24 466	9.25 146	0.74 854	9.99 319	53
8	9.20 067	9.20 621	0.79 379	9.99 446	9.24 536	9.25 219	0.74 781	9.99 317	52
9	9.20 145	9.20 701	0.79 299	9.99 444	9.24 607	9.25 292	0.74 708	9.99 315	51
10	9.20 223	9.20 782	0.79 218	9.99 442	9.24 677	9.25 365	0.74 635	9.99 313	**50**
11	9.20 302	9.20 862	0.79 138	9.99 440	9.24 748	9.25 437	0.74 563	9.99 310	49
12	9.20 380	9.20 942	0.79 058	9.99 438	9.24 818	9.25 510	0.74 490	9.99 308	48
13	9.20 458	9.21 022	0.78 978	9.99 436	9.24 888	9.25 582	0.74 418	9.99 306	47
14	9.20 535	9.21 102	0.78 898	9.99 434	9.24 958	9.25 655	0.74 345	9.99 304	46
15	9.20 613	9.21 182	0.78 818	9.99 432	9.25 028	9.25 727	0.74 273	9.99 301	45
16	9.20 691	9.21 261	0.78 739	9.99 429	9.25 098	9.25 799	0.74 201	9.99 299	44
17	9.20 768	9.21 341	0.78 659	9.99 427	9.25 168	9.25 871	0.74 129	9.99 297	43
18	9.20 845	9.21 420	0.78 580	9.99 425	9.25 237	9.25 943	0.74 057	9.99 294	42
19	9.20 922	9.21 499	0.78 501	9.99 423	9.25 307	9.26 015	0.73 985	9.99 292	41
20	9.20 999	9.21 578	0.78 422	9.99 421	9.25 376	9.26 086	0.73 914	9.99 290	**40**
21	9.21 076	9.21 657	0.78 343	9.99 419	9.25 445	9.26 158	0.73 842	9.99 288	39
22	9.21 153	9.21 736	0.78 264	9.99 417	9.25 514	9.26 229	0.73 771	9.99 285	38
23	9.21 229	9.21 814	0.78 186	9.99 415	9.25 583	9.26 301	0.73 699	9.99 283	37
24	9.21 306	9.21 893	0.78 107	9.99 413	9.25 652	9.26 372	0.73 628	9.99 281	36
25	9.21 382	9.21 971	0.78 029	9.99 411	9.25 721	9.26 443	0.73 557	9.99 278	35
26	9.21 458	9.22 049	0.77 951	9.99 409	9.25 790	9.26 514	0.73 486	9.99 276	34
27	9.21 534	9.22 127	0.77 873	9.99 407	9.25 858	9.26 585	0.73 415	9.99 274	33
28	9.21 610	9.22 205	0.77 795	9.99 404	9.25 927	9.26 655	0.73 345	9.99 271	32
29	9.21 685	9.22 283	0.77 717	9.99 402	9.25 995	9.26 726	0.73 274	9.99 269	31
30	9.21 761	9.22 361	0.77 639	9.99 400	9.26 063	9.26 797	0.73 203	9.99 267	**30**
31	9.21 836	9.22 438	0.77 562	9.99 398	9.26 131	9.26 867	0.73 133	9.99 264	29
32	9.21 912	9.22 516	0.77 484	9.99 396	9.26 199	9.26 937	0.73 063	9.99 262	28
33	9.21 987	9.22 593	0.77 407	9.99 394	9.26 267	9.27 008	0.72 992	9.99 260	27
34	9.22 062	9.22 670	0.77 330	9.99 392	9.26 335	9.27 078	0.72 922	9.99 257	26
35	9.22 137	9.22 747	0.77 253	9.99 390	9.26 403	9.27 148	0.72 852	9.99 255	25
36	9.22 211	9.22 824	0.77 176	9.99 388	9.26 470	9.27 218	0.72 782	9.99 252	24
37	9.22 286	9.22 901	0.77 099	9.99 385	9.26 538	9.27 288	0.72 712	9.99 250	23
38	9.22 361	9.22 977	0.77 023	9.99 383	9.26 605	9.27 357	0.72 643	9.99 248	22
39	9.22 435	9.23 054	0.76 946	9.99 381	9.26 672	9.27 427	0.72 573	9.99 245	21
40	9.22 509	9.23 130	0.76 870	9.99 379	9.26 739	9.27 496	0.72 504	9.99 243	**20**
41	9.22 583	9.23 206	0.76 794	9.99 377	9.26 806	9.27 566	0.72 434	9.99 241	19
42	9.22 657	9.23 283	0.76 717	9.99 375	9.26 873	9.27 635	0.72 365	9.99 238	18
43	9.22 731	9.23 359	0.76 641	9.99 372	9.26 940	9.27 704	0.72 296	9.99 236	17
44	9.22 805	9.23 435	0.76 565	9.99 370	9.27 007	9.27 773	0.72 227	9.99 233	16
45	9.22 878	9.23 510	0.76 490	9.99 368	9.27 073	9.27 842	0.72 158	9.99 231	15
46	9.22 952	9.23 586	0.76 414	9.99 366	9.27 140	9.27 911	0.72 089	9.99 229	14
47	9.23 025	9.23 661	0.76 339	9.99 364	9.27 206	9.27 980	0.72 020	9.99 226	13
48	9.23 098	9.23 737	0.76 263	9.99 362	9.27 273	9.28 049	0.71 951	9.99 224	12
49	9.23 171	9.23 812	0.76 188	9.99 359	9.27 339	9.28 117	0.71 883	9.99 221	11
50	9.23 244	9.23 887	0.76 113	9.99 357	9.27 405	9.28 186	0.71 814	9.99 219	**10**
51	9.23 317	9.23 962	0.76 038	9.99 355	9.27 471	9.28 254	0.71 746	9.99 217	9
52	9.23 390	9.24 037	0.75 963	9.99 353	9.27 537	9.28 323	0.71 677	9.99 214	8
53	9.23 462	9.24 112	0.75 888	9.99 351	9.27 602	9.28 391	0.71 609	9.99 212	7
54	9.23 535	9.24 186	0.75 814	9.99 348	9.27 668	9.28 459	0.71 541	9.99 209	6
55	9.23 607	9.24 261	0.75 739	9.99 346	9.27 734	9.28 527	0.71 473	9.99 207	5
56	9.23 679	9.24 335	0.75 665	9.99 344	9.27 799	9.28 595	0.71 405	9.99 204	4
57	9.23 752	9.24 410	0.75 590	9.99 342	9.27 864	9.28 662	0.71 338	9.99 202	3
58	9.23 823	9.24 484	0.75 516	9.99 340	9.27 930	9.28 730	0.71 270	9.99 200	2
59	9.23 895	9.24 558	0.75 442	9.99 337	9.27 995	9.28 798	0.71 202	9.99 197	1
60	9.23 967	9.24 632	0.75 368	9.99 335	9.28 060	9.28 865	0.71 135	9.99 195	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	80°				79°				′

′	11°				12°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.28 060	9.28 865	0.71 135	9.99 195	9.31 788	9.32 747	0.67 253	9.99 040	**60**
1	9.28 125	9.28 933	0.71 067	9.99 192	9.31 847	9.32 810	0.67 190	9.99 038	59
2	9.28 190	9.29 000	0.71 000	9.99 190	9.31 907	9.32 872	0.67 128	9.99 035	58
3	9.28 254	9.29 067	0.70 933	9.99 187	9.31 966	9.32 933	0.67 067	9.99 032	57
4	9.28 319	9.29 134	0.70 866	9.99 185	9.32 025	9.32 995	0.67 005	9.99 030	56
5	9.28 384	9.29 201	0.70 799	9.99 182	9.32 084	9.33 057	0.66 943	9.99 027	55
6	9.28 448	9.29 268	0.70 732	9.99 180	9.32 143	9.33 119	0.66 881	9.99 024	54
7	9.28 512	9.29 335	0.70 665	9.99 177	9.32 202	9.33 180	0.66 820	9.99 022	53
8	9.28 577	9.29 402	0.70 598	9.99 175	9.32 261	9.33 242	0.66 758	9.99 019	52
9	9.28 641	9.29 468	0.70 532	9.99 172	9.32 319	9.33 303	0.66 697	9.99 016	51
10	9.28 705	9.29 535	0.70 465	9.99 170	9.32 378	9.33 365	0.66 635	9.99 013	**50**
11	9.28 769	9.29 601	0.70 399	9.99 167	9.32 437	9.33 426	0.66 574	9.99 011	49
12	9.28 833	9.29 668	0.70 332	9.99 165	9.32 495	9.33 487	0.66 513	9.99 008	48
13	9.28 896	9.29 734	0.70 266	9.99 162	9.32 553	9.33 548	0.66 452	9.99 005	47
14	9.28 960	9.29 800	0.70 200	9.99 160	9.32 612	9.33 609	0.66 391	9.99 002	46
15	9.29 024	9.29 866	0.70 134	9.99 157	9.32 670	9.33 670	0.66 330	9.99 000	45
16	9.29 087	9.29 932	0.70 068	9.99 155	9.32 728	9.33 731	0.66 269	9.98 997	44
17	9.29 150	9.29 998	0.70 002	9.99 152	9.32 786	9.33 792	0.66 208	9.98 994	43
18	9.29 214	9.30 064	0.69 936	9.99 150	9.32 844	9.33 853	0.66 147	9.98 991	42
19	9.29 277	9.30 130	0.69 870	9.99 147	9.32 902	9.33 913	0.66 087	9.98 989	41
20	9.29 340	9.30 195	0.69 805	9.99 145	9.32 960	9.33 974	0.66 026	9.98 986	**40**
21	9.29 403	9.30 261	0.69 739	9.99 142	9.33 018	9.34 034	0.65 966	9.98 983	39
22	9.29 466	9.30 326	0.69 674	9.99 140	9.33 075	9.34 095	0.65 905	9.98 980	38
23	9.29 529	9.30 391	0.69 609	9.99 137	9.33 133	9.34 155	0.65 845	9.98 978	37
24	9.29 591	9.30 457	0.69 543	9.99 135	9.33 190	9.34 215	0.65 785	9.98 975	36
25	9.29 654	9.30 522	0.69 478	9.99 132	9.33 248	9.34 276	0.65 724	9.98 972	35
26	9.29 716	9.30 587	0.69 413	9.99 130	9.33 305	9.34 336	0.65 664	9.98 969	34
27	9.29 779	9.30 652	0.69 348	9.99 127	9.33 362	9.34 396	0.65 604	9.98 967	33
28	9.29 841	9.30 717	0.69 283	9.99 124	9.33 420	9.34 456	0.65 544	9.98 964	32
29	9.29 903	9.30 782	0.69 218	9.99 122	9.33 477	9.34 516	0.65 484	9.98 961	31
30	9.29 966	9.30 846	0.69 154	9.99 119	9.33 534	9.34 576	0.65 424	9.98 958	**30**
31	9.30 028	9.30 911	0.69 089	9.99 117	9.33 591	9.34 635	0.65 365	9.98 955	29
32	9.30 090	9.30 975	0.69 025	9.99 114	9.33 647	9.34 695	0.65 305	9.98 953	28
33	9.30 151	9.31 040	0.68 960	9.99 112	9.33 704	9.34 755	0.65 245	9.98 950	27
34	9.30 213	9.31 104	0.68 896	9.99 109	9.33 761	9.34 814	0.65 186	9.98 947	26
35	9.30 275	9.31 168	0.68 832	9.99 106	9.33 818	9.34 874	0.65 126	9.98 944	25
36	9.30 336	9.31 233	0.68 767	9.99 104	9.33 874	9.34 933	0.65 067	9.98 941	24
37	9.30 398	9.31 297	0.68 703	9.99 101	9.33 931	9.34 992	0.65 008	9.98 938	23
38	9.30 459	9.31 361	0.68 639	9.99 099	9.33 987	9.35 051	0.64 949	9.98 936	22
39	9.30 521	9.31 425	0.68 575	9.99 096	9.34 043	9.35 111	0.64 889	9.98 933	21
40	9.30 582	9.31 489	0.68 511	9.99 093	9.34 100	9.35 170	0.64 830	9.98 930	**20**
41	9.30 643	9.31 552	0.68 448	9.99 091	9.34 156	9.35 229	0.64 771	9.98 927	19
42	9.30 704	9.31 616	0.68 384	9.99 088	9.34 212	9.35 288	0.64 712	9.98 924	18
43	9.30 765	9.31 679	0.68 321	~~9.99~~ 086	9.34 268	9.35 347	0.64 653	9.98 921	17
44	9.30 826	9.31 743	0.68 257	9.99 083	9.34 324	9.35 405	0.64 595	9.98 919	16
45	9.30 887	9.31 806	0.68 194	9.99 080	9.34 380	9.35 464	0.64 536	9.98 916	15
46	9.30 947	9.31 870	0.68 130	9.99 078	9.34 436	9.35 523	0.64 477	9.98 913	14
47	9.31 008	9.31 933	0.68 067	9.99 075	9.34 491	9.35 581	0.64 419	9.98 910	13
48	9.31 068	9.31 996	0.68 004	9.99 072	9.34 547	9.35 640	0.64 360	9.98 907	12
49	9.31 129	9.32 059	0.67 941	9.99 070	9.34 602	9.35 698	0.64 302	9.98 904	11
50	9.31 189	9.32 122	0.67 878	9.99 067	9.34 658	9.35 757	0.64 243	9.98 901	**10**
51	9.31 250	9.32 185	0.67 815	9.99 064	9.34 713	9.35 815	0.64 185	9.98 898	9
52	9.31 310	9.32 248	0.67 752	9.99 062	9.34 769	9.35 873	0.64 127	9.98 896	8
53	9.31 370	9.32 311	0.67 689	9.99 059	9.34 824	9.35 931	0.64 069	9.98 893	7
54	9.31 430	9.32 373	0.67 627	9.99 056	9.34 879	9.35 989	0.64 011	9.98 890	6
55	9.31 490	9.32 436	0.67 564	9.99 054	9.34 934	9.36 047	0.63 953	9.98 887	5
56	9.31 549	9.32 498	0.67 502	9.99 051	9.34 989	9.36 105	0.63 895	9.98 884	4
57	9.31 609	9.32 561	0.67 439	9.99 048	9.35 044	9.36 163	0.63 837	9.98 881	3
58	9.31 669	9.32 623	0.67 377	9.99 046	9.35 099	9.36 221	0.63 779	9.98 878	2
59	9.31 728	9.32 685	0.67 315	9.99 043	9.35 154	9.36 279	0.63 721	9.98 875	1
60	9.31 788	9.32 747	0.67 253	9.99 040	9.35 209	9.36 336	0.63 664	9.98 872	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	78°				77°				′

′	13°				14°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.35 209	9.36 336	0.63 664	9.98 872	9.38 368	9.39 677	0.60 323	9.98 690	60
1	9.35 263	9.36 394	0.63 606	9.98 869	9.38 418	9.39 731	0.60 269	9.98 687	59
2	9.35 318	9.36 452	0.63 548	9.98 867	9.38 469	9.39 785	0.60 215	9.98 684	58
3	9.35 373	9.36 509	0.63 491	9.98 864	9.38 519	9.39 838	0.60 162	9.98 681	57
4	9.35 427	9.36 566	0.63 434	9.98 861	9.38 570	9.39 892	0.60 108	9.98 678	56
5	9.35 481	9.36 624	0.63 376	9.98 858	9.38 620	9.39 945	0.60 055	9.98 675	55
6	9.35 536	9.36 681	0.63 319	9.98 855	9.38 670	9.39 999	0.60 001	9.98 671	54
7	9.35 590	9.36 738	0.63 262	9.98 852	9.38 721	9.40 052	0.59 948	9.98 668	53
8	9.35 644	9.36 795	0.63 205	9.98 849	9.38 771	9.40 106	0.59 894	9.98 665	52
9	9.35 698	9.36 852	0.63 148	9.98 846	9.38 821	9.40 159	0.59 841	9.98 662	51
10	9.35 752	9.36 909	0.63 091	9.98 843	9.38 871	9.40 212	0.59 788	9.98 659	50
11	9.35 806	9.36 966	0.63 034	9.98 840	9.38 921	9.40 266	0.59 734	9.98 656	49
12	9.35 860	9.37 023	0.62 977	9.98 837	9.38 971	9.40 319	0.59 681	9.98 652	48
13	9.35 914	9.37 080	0.62 920	9.98 834	9.39 021	9.40 372	0.59 628	9.98 649	47
14	9.35 968	9.37 137	0.62 863	9.98 831	9.39 071	9.40 425	0.59 575	9.98 646	46
15	9.36 022	9.37 193	0.62 807	9.98 828	9.39 121	9.40 478	0.59 522	9.98 643	45
16	9.36 075	9.37 250	0.62 750	9.98 825	9.39 170	9.40 531	0.59 469	9.98 640	44
17	9.36 129	9.37 306	0.62 694	9.98 822	9.39 220	9.40 584	0.59 416	9.98 636	43
18	9.36 182	9.37 363	0.62 637	9.98 819	9.39 270	9.40 636	0.59 364	9.98 633	42
19	9.36 236	9.37 419	0.62 581	9.98 816	9.39 319	9.40 689	0.59 311	9.98 630	41
20	9.36 289	9.37 476	0.62 524	9.98 813	9.39 369	9.40 742	0.59 258	9.98 627	40
21	9.36 342	9.37 532	0.62 468	9.98 810	9.39 418	9.40 795	0.59 205	9.98 623	39
22	9.36 395	9.37 588	0.62 412	9.98 807	9.39 467	9.40 847	0.59 153	9.98 620	38
23	9.36 449	9.37 644	0.62 356	9.98 804	9.39 517	9.40 900	0.59 100	9.98 617	37
24	9.36 502	9.37 700	0.62 300	9.98 801	9.39 566	9.40 952	0.59 048	9.98 614	36
25	9.36 555	9.37 756	0.62 244	9.98 798	9.39 615	9.41 005	0.58 995	9.98 610	35
26	9.36 608	9.37 812	0.62 188	9.98 795	9.39 664	9.41 057	0.58 943	9.98 607	34
27	9.36 660	9.37 868	0.62 132	9.98 792	9.39 713	9.41 109	0.58 891	9.98 604	33
28	9.36 713	9.37 924	0.62 076	9.98 789	9.39 762	9.41 161	0.58 839	9.98 601	32
29	9.36 766	9.37 980	0.62 020	9.98 786	9.39 811	9.41 214	0.58 786	9.98 597	31
30	9.36 819	9.38 035	0.61 965	9.98 783	9.39 860	9.41 266	0.58 734	9.98 594	30
31	9.36 871	9.38 091	0.61 909	9.98 780	9.39 909	9.41 318	0.58 682	9.98 591	29
32	9.36 924	9.38 147	0.61 853	9.98 777	9.39 958	9.41 370	0.58 630	9.98 588	28
33	9.36 976	9.38 202	0.61 798	9.98 774	9.40 006	9.41 422	0.58 578	9.98 584	27
34	9.37 028	9.38 257	0.61 743	9.98 771	9.40 055	9.41 474	0.58 526	9.98 581	26
35	9.37 081	9.38 313	0.61 687	9.98 768	9.40 103	9.41 526	0.58 474	9.98 578	25
36	9.37 133	9.38 368	0.61 632	9.98 765	9.40 152	9.41 578	0.58 422	9.98 574	24
37	9.37 185	9.38 423	0.61 577	9.98 762	9.40 200	9.41 629	0.58 371	9.98 571	23
38	9.37 237	9.38 479	0.61 521	9.98 759	9.40 249	9.41 681	0.58 319	9.98 568	22
39	9.37 289	9.38 534	0.61 466	9.98 756	9.40 297	9.41 733	0.58 267	9.98 565	21
40	9.37 341	9.38 589	0.61 411	9.98 753	9.40 346	9.41 784	0.58 216	9.98 561	20
41	9.37 393	9.38 644	0.61 356	9.98 750	9.40 394	9.41 836	0.58 164	9.98 558	19
42	9.37 445	9.38 699	0.61 301	9.98 746	9.40 442	9.41 887	0.58 113	9.98 555	18
43	9.37 497	9.38 754	0.61 246	9.98 743	9.40 490	9.41 939	0.58 061	9.98 551	17
44	9.37 549	9.38 808	0.61 192	9.98 740	9.40 538	9.41 990	0.58 010	9.98 548	16
45	9.37 600	9.38 863	0.61 137	9.98 737	9.40 586	9.42 041	0.57 959	9.98 545	15
46	9.37 652	9.38 918	0.61 082	9.98 734	9.40 634	9.42 093	0.57 907	9.98 541	14
47	9.37 703	9.38 972	0.61 028	9.98 731	9.40 682	9.42 144	0.57 856	9.98 538	13
48	9.37 755	9.39 027	0.60 973	9.98 728	9.40 730	9.42 195	0.57 805	9.98 535	12
49	9.37 806	9.39 082	0.60 918	9.98 725	9.40 778	9.42 246	0.57 754	9.98 531	11
50	9.37 858	9.39 136	0.60 864	9.98 722	9.40 825	9.42 297	0.57 703	9.98 528	10
51	9.37 909	9.39 190	0.60 810	9.98 719	9.40 873	9.42 348	0.57 652	9.98 525	9
52	9.37 960	9.39 245	0.60 755	9.98 715	9.40 921	9.42 399	0.57 601	9.98 521	8
53	9.38 011	9.39 299	0.60 701	9.98 712	9.40 968	9.42 450	0.57 550	9.98 518	7
54	9.38 062	9.39 353	0.60 647	9.98 709	9.41 016	9.42 501	0.57 499	9.98 515	6
55	9.38 113	9.39 407	0.60 593	9.98 706	9.41 063	9.42 552	0.57 448	9.98 511	5
56	9.38 164	9.39 461	0.60 539	9.98 703	9.41 111	9.42 603	0.57 397	9.98 508	4
57	9.38 215	9.39 515	0.60 485	9.98 700	9.41 158	9.42 653	0.57 347	9.98 505	3
58	9.38 266	9.39 569	0.60 431	9.98 697	9.41 205	9.42 704	0.57 296	9.98 501	2
59	9.38 317	9.39 623	0.60 377	9.98 694	9.41 252	9.42 755	0.57 245	9.98 498	1
60	9.38 368	9.39 677	0.60 323	9.98 690	9.41 300	9.42 805	0.57 195	9.98 494	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	76°				75°				′

′	15°				16°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.41 300	9.42 805	0.57 195	9.98 494	9.44 034	9.45 750	0.54 250	9.98 284	**60**
1	9.41 347	9.42 856	0.57 144	9.98 491	9.44 078	9.45 797	0.54 203	9.98 281	59
2	9.41 394	9.42 906	0.57 094	9.98 488	9.44 122	9.45 845	0.54 155	9.98 277	58
3	9.41 441	9.42 957	0.57 043	9.98 484	9.44 166	9.45 892	0.54 108	9.98 273	57
4	9.41 488	9.43 007	0.56 993	9.98 481	9.44 210	9.45 940	0.54 060	9.98 270	56
5	9.41 535	9.43 057	0.56 943	9.98 477	9.44 253	9.45 987	0.54 013	9.98 266	55
6	9.41 582	9.43 108	0.56 892	9.98 474	9.44 297	9.46 035	0.53 965	9.98 262	54
7	9.41 628	9.43 158	0.56 842	9.98 471	9.44 341	9.46 082	0.53 918	9.98 259	53
8	9.41 675	9.43 208	0.56 792	9.98 467	9.44 385	9.46 130	0.53 870	9.98 255	52
9	9.41 722	9.43 258	0.56 742	9.98 464	9.44 428	9.46 177	0.53 823	9.98 251	51
10	9.41 768	9.43 308	0.56 692	9.98 460	9.44 472	9.46 224	0.53 776	9.98 248	**50**
11	9.41 815	9.43 358	0.56 642	9.98 457	9.44 516	9.46 271	0.53 729	9.98 244	49
12	9.41 861	9.43 408	0.56 592	9.98 453	9.44 559	9.46 319	0.53 681	9.98 240	48
13	9.41 908	9.43 458	0.56 542	9.98 450	9.44 602	9.46 366	0.53 634	9.98 237	47
14	9.41 954	9.43 508	0.56 492	9.98 447	9.44 646	9.46 413	0.53 587	9.98 233	46
15	9.42 001	9.43 558	0.56 442	9.98 443	9.44 689	9.46 460	0.53 540	9.98 229	45
16	9.42 047	9.43 607	0.56 393	9.98 440	9.44 733	9.46 507	0.53 493	9.98 226	44
17	9.42 093	9.43 657	0.56 343	9.98 436	9.44 776	9.46 554	0.53 446	9.98 222	43
18	9.42 140	9.43 707	0.56 293	9.98 433	9.44 819	9.46 601	0.53 399	9.98 218	42
19	9.42 186	9.43 756	0.56 244	9.98 429	9.44 862	9.46 648	0.53 352	9.98 215	41
20	9.42 232	9.43 806	0.56 194	9.98 426	9.44 905	9.46 694	0.53 306	9.98 211	**40**
21	9.42 278	9.43 855	0.56 145	9.98 422	9.44 948	9.46 741	0.53 259	9.98 207	39
22	9.42 324	9.43 905	0.56 095	9.98 419	9.44 992	9.46 788	0.53 212	9.98 204	38
23	9.42 370	9.43 954	0.56 046	9.98 415	9.45 035	9.46 835	0.53 165	9.98 200	37
24	9.42 416	9.44 004	0.55 996	9.98 412	9.45 077	9.46 881	0.53 119	9.98 196	36
25	9.42 461	9.44 053	0.55 947	9.98 409	9.45 120	9.46 928	0.53 072	9.98 192	35
26	9.42 507	9.44 102	0.55 898	9.98 405	9.45 163	9.46 975	0.53 025	9.98 189	34
27	9.42 553	9.44 151	0.55 849	9.98 402	9.45 206	9.47 021	0.52 979	9.98 185	33
28	9.42 599	9.44 201	0.55 799	9.98 398	9.45 249	9.47 068	0.52 932	9.98 181	32
29	9.42 644	9.44 250	0.55 750	9.98 395	9.45 292	9.47 114	0.52 886	9.98 177	31
30	9.42 690	9.44 299	0.55 701	9.98 391	9.45 334	9.47 160	0.52 840	9.98 174	**30**
31	9.42 735	9.44 348	0.55 652	9.98 388	9.45 377	9.47 207	0.52 793	9.98 170	29
32	9.42 781	9.44 397	0.55 603	9.98 384	9.45 419	9.47 253	0.52 747	9.98 166	28
33	9.42 826	9.44 446	0.55 554	9.98 381	9.45 462	9.47 299	0.52 701	9.98 162	27
34	9.42 872	9.44 495	0.55 505	9.98 377	9.45 504	9.47 346	0.52 654	9.98 159	26
35	9.42 917	9.44 544	0.55 456	9.98 373	9.45 547	9.47 392	0.52 608	9.98 155	25
36	9.42 962	9.44 592	0.55 408	9.98 370	9.45 589	9.47 438	0.52 562	9.98 151	24
37	9.43 008	9.44 641	0.55 359	9.98 366	9.45 632	9.47 484	0.52 516	9.98 147	23
38	9.43 053	9.44 690	0.55 310	9.98 363	9.45 674	9.47 530	0.52 470	9.98 144	22
39	9.43 098	9.44 738	0.55 262	9.98 359	9.45 716	9.47 576	0.52 424	9.98 140	21
40	9.43 143	9.44 787	0.55 213	9.98 356	9.45 758	9.47 622	0.52 378	9.98 136	**20**
41	9.43 188	9.44 836	0.55 164	9.98 352	9.45 801	9.47 668	0.52 332	9.98 132	19
42	9.43 233	9.44 884	0.55 116	9.98 349	9.45 843	9.47 714	0.52 286	9.98 129	18
43	9.43 278	9.44 933	0.55 067	9.98 345	9.45 885	9.47 760	0.52 240	9.98 125	17
44	9.43 323	9.44 981	0.55 019	9.98 342	9.45 927	9.47 806	0.52 194	9.98 121	16
45	9.43 367	9.45 029	0.54 971	9.98 338	9.45 969	9.47 852	0.52 148	9.98 117	15
46	9.43 412	9.45 078	0.54 922	9.98 334	9.46 011	9.47 897	0.52 103	9.98 113	14
47	9.43 457	9.45 126	0.54 874	9.98 331	9.46 053	9.47 943	0.52 057	9.98 110	13
48	9.43 502	9.45 174	0.54 826	9.98 327	9.46 095	9.47 989	0.52 011	9.98 106	12
49	9.43 546	9.45 222	0.54 778	9.98 324	9.46 136	9.48 035	0.51 965	9.98 102	11
50	9.43 591	9.45 271	0.54 729	9.98 320	9.46 178	9.48 080	0.51 920	9.98 098	**10**
51	9.43 635	9.45 319	0.54 681	9.98 317	9.46 220	9.48 126	0.51 874	9.98 094	9
52	9.43 680	9.45 367	0.54 633	9.98 313	9.46 262	9.48 171	0.51 829	9.98 090	8
53	9.43 724	9.45 415	0.54 585	9.98 309	9.46 303	9.48 217	0.51 783	9.98 087	7
54	9.43 769	9.45 463	0.54 537	9.98 306	9.46 345	9.48 262	0.51 738	9.98 083	6
55	9.43 813	9.45 511	0.54 489	9.98 302	9.46 386	9.48 307	0.51 693	9.98 079	5
56	9.43 857	9.45 559	0.54 441	9.98 299	9.46 428	9.48 353	0.51 647	9.98 075	4
57	9.43 901	9.45 606	0.54 394	9.98 295	9.46 469	9.48 398	0.51 602	9.98 071	3
58	9.43 946	9.45 654	0.54 346	9.98 291	9.46 511	9.48 443	0.51 557	9.98 067	2
59	9.43 990	9.45 702	0.54 298	9.98 288	9.46 552	9.48 489	0.51 511	9.98 063	1
60	9.44 034	9.45 750	0.54 250	9.98 284	9.46 594	9.48 534	0.51 466	9.98 060	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	74°				73°				′

′	17°				18°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.46 594	9.48 534	0.51 466	9.98 060	9.48 998	9.51 178	0.48 822	9.97 821	**60**
1	9.46 635	9.48 579	0.51 421	9.98 056	9.49 037	9.51 221	0.48 779	9.97 817	59
2	9.46 676	9.48 624	0.51 376	9.98 052	9.49 076	9.51 264	0.48 736	9.97 812	58
3	9.46 717	9.48 669	0.51 331	9.98 048	9.49 115	9.51 306	0.48 694	9.97 808	57
4	9.46 758	9.48 714	0.51 286	9.98 044	9.49 153	9.51 349	0.48 651	9.97 804	56
5	9.46 800	9.48 759	0.51 241	9.98 040	9.49 192	9.51 392	0.48 608	9.97 800	55
6	9.46 841	9.48 804	0.51 196	9.98 036	9.49 231	9.51 435	0.48 565	9.97 796	54
7	9.46 882	9.48 849	0.51 151	9.98 032	9.49 269	9.51 478	0.48 522	9.97 792	53
8	9.46 923	9.48 894	0.51 106	9.98 029	9.49 308	9.51 520	0.48 480	9.97 788	52
9	9.46 964	9.48 939	0.51 061	9.98 025	9.49 347	9.51 563	0.48 437	9.97 784	51
10	9.47 005	9.48 984	0.51 016	9.98 021	9.49 385	9.51 606	0.48 394	9.97 779	**50**
11	9.47 045	9.49 029	0.50 971	9.98 017	9.49 424	9.51 648	0.48 352	9.97 775	49
12	9.47 086	9.49 073	0.50 927	9.98 013	9.49 462	9.51 691	0.48 309	9.97 771	48
13	9.47 127	9.49 118	0.50 882	9.98 009	9.49 500	9.51 734	0.48 266	9.97 767	47
14	9.47 168	9.49 163	0.50 837	9.98 005	9.49 539	9.51 776	0.48 224	9.97 763	46
15	9.47 209	9.49 207	0.50 793	9.98 001	9.49 577	9.51 819	0.48 181	9.97 759	45
16	9.47 249	9.49 252	0.50 748	9.97 997	9.49 615	9.51 861	0.48 139	9.97 754	44
17	9.47 290	9.49 296	0.50 704	9.97 993	9.49 654	9.51 903	0.48 097	9.97 750	43
18	9.47 330	9.49 341	0.50 659	9.97 989	9.49 692	9.51 946	0.48 054	9.97 746	42
19	9.47 371	9.49 385	0.50 615	9.97 986	9.49 730	9.51 988	0.48 012	9.97 742	41
20	9.47 411	9.49 430	0.50 570	9.97 982	9.49 768	9.52 031	0.47 969	9.97 738	**40**
21	9.47 452	9.49 474	0.50 526	9.97 978	9.49 806	9.52 073	0.47 927	9.97 734	39
22	9.47 492	9.49 519	0.50 481	9.97 974	9.49 844	9.52 115	0.47 885	9.97 729	38
23	9.47 533	9.49 563	0.50 437	9.97 970	9.49 882	9.52 157	0.47 843	9.97 725	37
24	9.47 573	9.49 607	0.50 393	9.97 966	9.49 920	9.52 200	0.47 800	9.97 721	36
25	9.47 613	9.49 652	0.50 348	9.97 962	9.49 958	9.52 242	0.47 758	9.97 717	35
26	9.47 654	9.49 696	0.50 304	9.97 958	9.49 996	9.52 284	0.47 716	9.97 713	34
27	9.47 694	9.49 740	0.50 260	9.97 954	9.50 034	9.52 326	0.47 674	9.97 708	33
28	9.47 734	9.49 784	0.50 216	9.97 950	9.50 072	9.52 368	0.47 632	9.97 704	32
29	9.47 774	9.49 828	0.50 172	9.97 946	9.50 110	9.52 410	0.47 590	9.97 700	31
30	9.47 814	9.49 872	0.50 128	9.97 942	9.50 148	9.52 452	0.47 548	9.97 696	**30**
31	9.47 854	9.49 916	0.50 084	9.97 938	9.50 185	9.52 494	0.47 506	9.97 691	29
32	9.47 894	9.49 960	0.50 040	9.97 934	9.50 223	9.52 536	0.47 464	9.97 687	28
33	9.47 934	9.50 004	0.49 996	9.97 930	9.50 261	9.52 578	0.47 422	9.97 683	27
34	9.47 974	9.50 048	0.49 952	9.97 926	9.50 298	9.52 620	0.47 380	9.97 679	26
35	9.48 014	9.50 092	0.49 908	9.97 922	9.50 336	9.52 661	0.47 339	9.97 674	25
36	9.48 054	9.50 136	0.49 864	9.97 918	9.50 374	9.52 703	0.47 297	9.97 670	24
37	9.48 094	9.50 180	0.49 820	9.97 914	9.50 411	9.52 745	0.47 255	9.97 666	23
38	9.48 133	9.50 223	0.49 777	9.97 910	9.50 449	9.52 787	0.47 213	9.97 662	22
39	9.48 173	9.50 267	0.49 733	9.97 906	9.50 486	9.52 829	0.47 171	9.97 657	21
40	9.48 213	9.50 311	0.49 689	9.97 902	9.50 523	9.52 870	0.47 130	9.97 653	**20**
41	9.48 252	9.50 355	0.49 645	9.97 898	9.50 561	9.52 912	0.47 088	9.97 649	19
42	9.48 292	9.50 398	0.49 602	9.97 894	9.50 598	9.52 953	0.47 047	9.97 645	18
43	9.48 332	9.50 442	0.49 558	9.97 890	9.50 635	9.52 995	0.47 005	9.97 640	17
44	9.48 371	9.50 485	0.49 515	9.97 886	9.50 673	9.53 037	0.46 963	9.97 636	16
45	9.48 411	9.50 529	0.49 471	9.97 882	9.50 710	9.53 078	0.46 922	9.97 632	15
46	9.48 450	9.50 572	0.49 428	9.97 878	9.50 747	9.53 120	0.46 880	9.97 628	14
47	9.48 490	9.50 616	0.49 384	9.97 874	9.50 784	9.53 161	0.46 839	9.97 623	13
48	9.48 529	9.50 659	0.49 341	9.97 870	9.50 821	9.53 202	0.46 798	9.97 619	12
49	9.48 568	9.50 703	0.49 297	9.97 866	9.50 858	9.53 244	0.46 756	9.97 615	11
50	9.48 607	9.50 746	0.49 254	9.97 861	9.50 896	9.53 285	0.46 715	9.97 610	**10**
51	9.48 647	9.50 789	0.49 211	9.97 857	9.50 933	9.53 327	0.46 673	9.97 606	9
52	9.48 686	9.50 833	0.49 167	9.97 853	9.50 970	9.53 368	0.46 632	9.97 602	8
53	9.48 725	9.50 876	0.49 124	9.97 849	9.51 007	9.53 409	0.46 591	9.97 597	7
54	9.48 764	9.50 919	0.49 081	9.97 845	9.51 043	9.53 450	0.46 550	9.97 593	6
55	9.48 803	9.50 962	0.49 038	9.97 841	9.51 080	9.53 492	0.46 508	9.97 589	5
56	9.48 842	9.51 005	0.48 995	9.97 837	9.51 117	9.53 533	0.46 467	9.97 584	4
57	9.48 881	9.51 048	0.48 952	9.97 833	9.51 154	9.53 574	0.46 426	9.97 580	3
58	9.48 920	9.51 092	0.48 908	9.97 829	9.51 191	9.53 615	0.46 385	9.97 576	2
59	9.48 959	9.51 135	0.48 865	9.97 825	9.51 227	9.53 656	0.46 344	9.97 571	1
60	9.48 998	9.51 178	0.48 822	9.97 821	9.51 264	9.53 697	0.46 303	9.97 567	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	72°				71°				′

′	19°				20°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.51 264	9.53 697	0.46 303	9.97 567	9.53 405	9.56 107	0.43 893	9.97 299	60
1	9.51 301	9.53 738	0.46 262	9.97 563	9.53 440	9.56 146	0.43 854	9.97 294	59
2	9.51 338	9.53 779	0.46 221	9.97 558	9.53 475	9.56 185	0.43 815	9.97 289	58
3	9.51 374	9.53 820	0.46 180	9.97 554	9.53 509	9.56 224	0.43 776	9.97 285	57
4	9.51 411	9.53 861	0.46 139	9.97 550	9.53 544	9.56 264	0.43 736	9.97 280	56
5	9.51 447	9.53 902	0.46 098	9.97 545	9.53 578	9.56 303	0.43 697	9.97 276	55
6	9.51 484	9.53 943	0.46 057	9.97 541	9.53 613	9.56 342	0.43 658	9.97 271	54
7	9.51 520	9.53 984	0.46 016	9.97 536	9.53 647	9.56 381	0.43 619	9.97 266	53
8	9.51 557	9.54 025	0.45 975	9.97 532	9.53 682	9.56 420	0.43 580	9.97 262	52
9	9.51 593	9.54 065	0.45 935	9.97 528	9.53 716	9.56 459	0.43 541	9.97 257	51
10	9.51 629	9.54 106	0.45 894	9.97 523	9.53 751	9.56 498	0.43 502	9.97 252	50
11	9.51 666	9.54 147	0.45 853	9.97 519	9.53 785	9.66 537	0.43 463	9.97 248	49
12	9.51 702	9.54 187	0.45 813	9.97 515	9.53 819	9.56 576	0.43 424	9.97 243	48
13	9.51 738	9.54 228	0.45 772	9.97 510	9.53 854	9.56 615	0.43 385	9.97 238	47
14	9.51 774	9.54 269	0.45 731	9.97 506	9.53 888	9.56 654	0.43 346	9.97 234	46
15	9.51 811	9.54 309	0.45 691	9.97 501	9.53 922	9.56 693	0.43 307	9.97 229	45
16	9.51 847	9.54 350	0.45 650	9.97 497	9.53 957	9.56 732	0.43 268	9.97 224	44
17	9.51 883	9.54 390	0.45 610	9.97 492	9.53 991	9.56 771	0.43 229	9.97 220	43
18	9.51 919	9.54 431	0.45 569	9.97 488	9.54 025	9.56 810	0.43 190	9.97 215	42
19	9.51 955	9.54 471	0.45 529	9.97 484	9.54 059	9.56 849	0.43 151	9.97 210	41
20	9.51 991	9.54 512	0.45 488	9.97 479	9.54 093	9.56 887	0.43 113	9.97 206	40
21	9.52 027	9.54 552	0.45 448	9.97 475	9.54 127	9.56 926	0.43 074	9.97 201	39
22	9.52 063	9.54 593	0.45 407	9.97 470	9.54 161	9.56 965	0.43 035	9.97 196	38
23	9.52 099	9.54 633	0.45 367	9.97 466	9.54 195	9.57 004	0.42 996	9.97 192	37
24	9.52 135	9.54 673	0.45 327	9.97 461	9.54 229	9.57 042	0.42 958	9.97 187	36
25	9.52 171	9.54 714	0.45 286	9.97 457	9.54 263	9.57 081	0.42 919	9.97 182	35
26	9.52 207	9.54 754	0.45 246	9.97 453	9.54 297	9.57 120	0.42 880	9.97 178	34
27	9.52 242	9.54 794	0.45 206	9.97 448	9.54 331	9.57 158	0.42 842	9.97 173	33
28	9.52 278	9.54 835	0.45 165	9.97 444	9.54 365	9.57 197	0.42 803	9.97 168	32
29	9.52 314	9.54 875	0.45 125	9.97 439	9.54 399	9.57 235	0.42 765	9.97 163	31
30	9.52 350	9.54 915	0.45 085	9.97 435	9.54 433	9.57 274	0.42 726	9.97 159	30
31	9.52 385	9.54 955	0.45 045	9.97 430	9.54 466	9.57 312	0.42 688	9.97 154	29
32	9.52 421	9.54 995	0.45 005	9.97 426	9.54 500	9.57 351	0.42 649	9.97 149	28
33	9.52 456	9.55 035	0.44 965	9.97 421	9.54 534	9.57 389	0.42 611	9.97 145	27
34	9.52 492	9.55 075	0.44 925	9.97 417	9.54 567	9.57 428	0.42 572	9.97 140	26
35	9.52 527	9.55 115	0.44 885	9.97 412	9.54 601	9.57 466	0.42 534	9.97 135	25
36	9.52 563	9.55 155	0.44 845	9.97 408	9.54 635	9.57 504	0.42 496	9.97 130	24
37	9.52 598	9.55 195	0.44 805	9.97 403	9.54 668	9.57 543	0.42 457	9.97 126	23
38	9.52 634	9.55 235	0.44 765	9.97 399	9.54 702	9.57 581	0.42 419	9.97 121	22
39	9.52 669	9.55 275	0.44 725	9.97 394	9.54 735	9.57 619	0.42 381	9.97 116	21
40	9.52 705	9.55 315	0.44 685	9.97 390	9.54 769	9.57 658	0.42 342	9.97 111	20
41	9.52 740	9.55 355	0.44 645	9.97 385	9.54 802	9.57 696	0.42 304	9.97 107	19
42	9.52 775	9.55 395	0.44 605	9.97 381	9.54 836	9.57 734	0.42 266	9.97 102	18
43	9.52 811	9.55 434	0.44 566	9.97 376	9.54 869	9.57 772	0.42 228	9.97 097	17
44	9.52 846	9.55 474	0.44 526	9.97 372	9.54 903	9.57 810	0.42 190	9.97 092	16
45	9.52 881	9.55 514	0.44 486	9.97 367	9.54 936	9.57 849	0.42 151	9.97 087	15
46	9.52 916	9.55 554	0.44 446	9.97 363	9.54 969	9.57 887	0.42 113	9.97 083	14
47	9.52 951	9.55 593	0.44 407	9.97 358	9.55 003	9.57 925	0.42 075	9.97 078	13
48	9.52 986	9.55 633	0.44 367	9.97 353	9.55 036	9.57 963	0.42 037	9.97 073	12
49	9.53 021	9.55 673	0.44 327	9.97 349	9.55 069	9.58 001	0.41 999	9.97 068	11
50	9.53 056	9.55 712	0.44 288	9.97 344	9.55 102	9.58 039	0.41 961	9.97 063	10
51	9.53 092	9.55 752	0.44 248	9.97 340	9.55 136	9.58 077	0.41 923	9.97 059	9
52	9.53 126	9.55 791	0.44 209	9.97 335	9.55 169	9.58 115	0.41 885	9.97 054	8
53	9.53 161	9.55 831	0.44 169	9.97 331	9.55 202	9.58 153	0.41 847	9.97 049	7
54	9.53 196	9.55 870	0.44 130	9.97 326	9.55 235	9.58 191	0.41 809	9.97 044	6
55	9.53 231	9.55 910	0.44 090	9.97 322	9.55 268	9.58 229	0.41 771	9.97 039	5
56	9.53 266	9.55 949	0.44 051	9.97 317	9.55 301	9.58 267	0.41 733	9.97 035	4
57	9.53 301	9.55 989	0.44 011	9.97 312	9.55 334	9.58 304	0.41 696	9.97 030	3
58	9.53 336	9.56 028	0.43 972	9.97 308	9.55 367	9.58 342	0.41 658	9.97 025	2
59	9.53 370	9.56 067	0.43 933	9.97 303	9.55 400	9.58 380	0.41 620	9.97 020	1
60	9.53 405	9.56 107	0.43 893	9.97 290	9.55 433	9.58 418	0.41 582	9.97 015	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	70°				69°				′

′	21°				22°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.55 433	9.58 418	0.41 582	9.97 015	9.57 358	9.60 641	0.39 359	9.96 717	**60**
1	9.55 466	9.58 455	0.41 545	9.97 010	9.57 389	9.60 677	0.39 323	9.96 711	59
2	9.55 499	9.58 493	0.41 507	9.97 005	9.57 420	9.60 714	0.39 286	9.96 706	58
3	9.55 532	9.58 531	0.41 469	9.97 001	9.57 451	9.60 750	0.39 250	9.96 701	57
4	9.55 564	9.58 569	0.41 431	9.96 996	9.57 482	9.60 786	0.39 214	9.96 696	56
5	9.55 597	9.58 606	0.41 394	9.96 991	9.57 514	9.60 823	0.39 177	9.96 691	55
6	9.55 630	9.58 644	0.41 356	9.96 986	9.57 545	9.60 859	0.39 141	9.96 686	54
7	9.55 663	9.58 681	0.41 319	9.96 981	9.57 576	9.60 895	0.39 105	9.96 681	53
8	9.55 695	9.58 719	0.41 281	9.96 976	9.57 607	9.60 931	0.39 069	9.96 676	52
9	9.55 728	9.58 757	0.41 243	9.96 971	9.57 638	9.60 967	0.39 033	9.96 670	51
10	9.55 761	9.58 794	0.41 206	9.96 966	9.57 669	9.61 004	0.38 996	9.96 665	**50**
11	9.55 793	9.58 832	0.41 168	9.96 962	9.57 700	9.61 040	0.38 960	9.96 660	49
12	9.55 826	9.58 869	0.41 131	9.96 957	9.57 731	9.61 076	0.38 924	9.96 655	48
13	9.55 858	9.58 907	0.41 093	9.96 952	9.57 762	9.61 112	0.38 888	9.96 650	47
14	9.55 891	9.58 944	0.41 056	9.96 947	9.57 793	9.61 148	0.38 852	9.96 645	46
15	9.55 923	9.58 981	0.41 019	9.96 942	9.57 824	9.61 184	0.38 816	9.96 640	45
16	9.55 956	9.59 019	0.40 981	9.96 937	9.57 855	9.61 220	0.38 780	9.96 634	44
17	9.55 988	9.59 056	0.40 944	9.96 932	9.57 885	9.61 256	0.38 744	9.96 629	43
18	9.56 021	9.59 094	0.40 906	9.96 927	9.57 916	9.61 292	0.38 708	9.96 624	42
19	9.56 053	9.59 131	0.40 869	9.96 922	9.57 947	9.61 328	0.38 672	9.96 619	41
20	9.56 085	9.59 168	0.40 832	9.96 917	9.57 978	9.61 364	0.38 636	9.96 614	**40**
21	9.56 118	9.59 205	0.40 795	9.96 912	9.58 008	9.61 400	0.38 600	9.96 608	39
22	9.56 150	9.59 243	0.40 757	9.96 907	9.58 039	9.61 436	0.38 564	9.96 603	38
23	9.56 182	9.59 280	0.40 720	9.96 903	9.58 070	9.61 472	0.38 528	9.96 598	37
24	9.56 215	9.59 317	0.40 683	9.96 898	9.58 101	9.61 508	0.38 492	9.96 593	36
25	9 56 247	9.59 354	0.40 646	9.96 893	9.58 131	9.61 544	0.38 456	9.96 588	35
26	9.56 279	9.59 391	0.40 609	9.96 888	9.58 162	9.61 579	0.38 421	9.96 582	34
27	9.56 311	9.59 429	0.40 571	9.96 883	9.58 192	9.61 615	0.38 385	9.96 577	33
28	9.56 343	9.59 466	0.40 534	9.96 878	9.58 223	9.61 651	0.38 349	9.96 572	32
29	9.56 375	9.59 503	0.40 497	9.96 873	9.58 253	9.61 687	0.38 313	9.96 567	31
30	9 56 408	9.59 540	0.40 460	9.96 868	9.58 284	9.61 722	0.38 278	9.96 562	**30**
31	9.56 440	9.59 577	0.40 423	9.96 863	9.58 314	9.61 758	0.38 242	9.96 556	29
32	9 56 472	9.59 614	0.40 386	9.96 858	9.58 345	9.61 794	0.38 206	9.96 551	28
33	9.56 504	9.59 651	0.40 349	9.96 853	9.58 375	9.61 830	0.38 170	9.96 546	27
34	9.56 536	9.59 688	0.40 312	9.96 848	9.58 406	9.61 865	0.38 135	9.96 541	26
35	9.56 568	9.59 725	0.40 275	9.96 843	9.58 436	9.61 901	0.38 099	9.96 535	25
36	9 56 599	9.59 762	0.40 238	9.96 838	9.58 467	9.61 936	0.38 064	9.96 530	24
37	9.56 631	9.59 799	0.40 201	9.96 833	9.58 497	9.61 972	0.38 028	9.96 525	23
38	9.56 663	9.59 835	0.40 165	9.96 828	9.58 527	9.62 008	0.37 992	9.96 520	22
39	9.56 695	9.59 872	0.40 128	9.96 823	9.58 557	9.62 043	0.37 957	9.96 514	21
40	9.56 727	9.59 909	0.40 091	9.96 818	9.58 588	9.62 079	0.37 921	9.96 509	**20**
41	9.56 759	9.59 946	0.40 054	9.96 813	9.58 618	9.62 114	0.37 886	9.96 504	19
42	9.56 790	9.59 983	0.40 017	9.96 808	9.58 648	9.62 150	0.37 850	9.96 498	18
43	9.56 822	9.60 019	0.39 981	9.96 803	9.58 678	9.62 185	0.37 815	9.96 493	17
44	9.56 854	9.60 056	0.39 944	9.96 798	9.58 709	9.62 221	0.37 779	9.96 488	16
45	9.56 886	9.60 093	0.39 907	9.96 793	9.58 739	9.62 256	0.37 744	9.96 483	15
46	9 56 917	9.60 130	0.39 870	9.96 788	9.58 769	9.62 292	0.37 708	9.96 477	14
47	9 56 949	9.60 166	0.39 834	9.96 783	9.58 799	9.62 327	0.37 673	9.96 472	13
48	9.56 980	9.60 203	0.39 797	9.96 778	9.58 829	9.62 362	0.37 638	9.96 467	12
49	9.57 012	9.60 240	0.39 760	9.96 772	9.58 859	9.62 398	0.37 602	9.96 461	11
50	9.57 044	9.60 276	0.39 724	9.96 767	9.58 889	9.62 433	0.37 567	9.96 456	**10**
51	9 57 075	9.60 313	0.39 687	9.96 762	9.58 919	9.62 468	0.37 532	9.96 451	9
52	9.57 107	9.60 349	0.39 651	9.96 757	9.58 949	9.62 504	0.37 496	9.96 445	8
53	9.57 138	9.60 386	0.39 614	9.96 752	9.58 979	9.62 539	0.37 461	9.96 440	7
54	9.57 169	9.60 422	0.39 578	9.96 747	9.59 009	9.62 574	0.37 426	9.96 435	6
55	9.57 201	9.60 459	0.39 541	9.96 742	9.59 039	9.62 609	0.37 391	9.96 429	5
56	9.57 232	9.60 495	0.39 505	9.96 737	9.59 069	9.62 645	0.37 355	9.96 424	4
57	9.57 264	9.60 532	0.39 468	9.96 732	9.59 098	9.62 680	0.37 320	9.96 419	3
58	9.57 295	9.60 568	0.39 432	9.96 727	9.59 128	9.62 715	0.37 285	9.96 413	2
59	9.57 326	9.60 605	0.39 395	9.96 722	9.59 158	9.62 750	0.37 250	9.96 408	1
60	9.57 358	9.60 641	0.39 359	9.96 717	9.59 188	9.62 785	0.37 215	9.96 403	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	68°				67°				′

′	23°				24°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.59 188	9.62 785	0.37 215	9.96 403	9.60 931	9.64 858	0.35 142	9.96 073	60
1	9.59 218	9.62 820	0.37 180	9.96 397	9.60 960	9.64 892	0.35 108	9.96 067	59
2	9.59 247	9.62 855	0.37 145	9.96 392	9.60 988	9.64 926	0.35 074	9.96 062	58
3	9.59 277	9.62 890	0.37 110	9.96 387	9.61 016	9.64 960	0.35 040	9.96 056	57
4	9.59 307	9.62 926	0.37 074	9.96 381	9.61 045	9.64 994	0.35 006	9.96 050	56
5	9.59 336	9.62 961	0.37 039	9.96 376	9.61 073	9.65 028	0.34 972	9.96 045	55
6	9.59 366	9.62 996	0.37 004	9.96 370	9.61 101	9.65 062	0.34 938	9.96 039	54
7	9.59 396	9.63 031	0.36 969	9.96 365	9.61 129	9.65 096	0.34 904	9.96 034	53
8	9.59 425	9.63 066	0.36 934	9.96 360	9.61 158	9.65 130	0.34 870	9.96 028	52
9	9.59 455	9.63 101	0.36 899	9.96 354	9.61 186	9.65 164	0.34 836	9.96 022	51
10	9.59 484	9.63 135	0.36 865	9.96 349	9.61 214	9.65 197	0.34 803	9.96 017	50
11	9.59 514	9.63 170	0.36 830	9.96 343	9.61 242	9.65 231	0.34 769	9.96 011	49
12	9.59 543	9.63 205	0.36 795	9.96 338	9.61 270	9.65 265	0.34 735	9.96 005	48
13	9.59 573	9.63 240	0.36 760	9.96 333	9.61 298	9.65 299	0.34 701	9.96 000	47
14	9.59 602	9.63 275	0.36 725	9.96 327	9.61 326	9.65 333	0.34 667	9.95 994	46
15	9.59 632	9.63 310	0.36 690	9.96 322	9.61 354	9.65 366	0.34 634	9.95 988	45
16	9.59 661	9.63 345	0.36 655	9.96 316	9.61 382	9.65 400	0.34 600	9.95 982	44
17	9.59 690	9.63 379	0.36 621	9.96 311	9.61 411	9.65 434	0.34 566	9.95 977	43
18	9.59 720	9.63 414	0.36 586	9.96 305	9.61 438	9.65 467	0.34 533	9.95 971	42
19	9.59 749	9.63 449	0.36 551	9.96 300	9.61 466	9.65 501	0.34 499	9.95 965	41
20	9.59 778	9.63 484	0.36 516	9.96 294	9.61 494	9.65 535	0.34 465	9.95 960	40
21	9.59 808	9.63 519	0.36 481	9.96 289	9.61 522	9.65 568	0.34 432	9.95 954	39
22	9.59 837	9.63 553	0.36 447	9.96 284	9.61 550	9.65 602	0.34 398	9.95 948	38
23	9.59 866	9.63 588	0.36 412	9.96 278	9.61 578	9.65 636	0.34 364	9.95 942	37
24	9.59 895	9.63 623	0.36 377	9.96 273	9.61 606	9.65 669	0.34 331	9.95 937	36
25	9.59 924	9.63 657	0.36 343	9.96 267	9.61 634	9.65 703	0.34 297	9.95 931	35
26	9.59 954	9.63 692	0.36 308	9.96 262	9.61 662	9.65 736	0.34 264	9.95 925	34
27	9.59 983	9.63 726	0.36 274	9.96 256	9.61 689	9.65 770	0.34 230	9.95 920	33
28	9.60 012	9.63 761	0.36 239	9.96 251	9.61 717	9.65 803	0.34 197	9.95 914	32
29	9.60 041	9.63 796	0.36 204	9.96 245	9.61 745	9.65 837	0.34 163	9.95 908	31
30	9.60 070	9.63 830	0.36 170	9.96 240	9.61 773	9.65 870	0.34 130	9.95 902	30
31	9.60 099	9.63 865	0.36 135	9.96 234	9.61 800	9.65 904	0.34 096	9.95 897	29
32	9.60 128	9.63 899	0.36 101	9.96 229	9.61 828	9.65 937	0.34 063	9.95 891	28
33	9.60 157	9.63 934	0.36 066	9.96 223	9.61 856	9.65 971	0.34 029	9.95 885	27
34	9.60 186	9.63 968	0.36 032	9.96 218	9.61 883	9.66 004	0.33 996	9.95 879	26
35	9.60 215	9.64 003	0.35 997	9.96 212	9.61 911	9.66 038	0.33 962	9.95 873	25
36	9.60 244	9.64 037	0.35 963	9.96 207	9.61 939	9.66 071	0.33 929	9.95 868	24
37	9.60 273	9.64 072	0.35 928	9.96 201	9.61 966	9.66 104	0.33 896	9.95 862	23
38	9.60 302	9.64 106	0.35 894	9.96 196	9.61 994	9.66 138	0.33 862	9.95 856	22
39	9.60 331	9.64 140	0.35 860	9.96 190	9.62 021	9.66 171	0.33 829	9.95 850	21
40	9.60 359	9.64 175	0.35 825	9.96 185	9.62 049	9.66 204	0.33 796	9.95 844	20
41	9.60 388	9.64 209	0.35 791	9.96 179	9.62 076	9.66 238	0.33 762	9.95 839	19
42	9.60 417	9.64 243	0.35 757	9.96 174	9.62 104	9.66 271	0.33 729	9.95 833	18
43	9.60 446	9.64 278	0.35 722	9.96 168	9.62 131	9.66 304	0.33 696	9.95 827	17
44	9.60 474	9.64 312	0.35 688	9.96 162	9.62 159	9.66 337	0.33 663	9.95 821	16
45	9.60 503	9.64 346	0.35 654	9.96 157	9.62 186	9.66 371	0.33 629	9.95 815	15
46	9.60 532	9.64 381	0.35 619	9.96 151	9.62 214	9.66 404	0.33 596	9.95 810	14
47	9.60 561	9.64 415	0.35 585	9.96 146	9.62 241	9.66 437	0.33 563	9.95 804	13
48	9.60 589	9.64 449	0.35 551	9.96 140	9.62 268	9.66 470	0.33 530	9.95 798	12
49	9.60 618	9.64 483	0.35 517	9.96 135	9.62 296	9.66 503	0.33 497	9.95 792	11
50	9.60 646	9.64 517	0.35 483	9.96 129	9.62 323	9.66 537	0.33 463	9.95 786	10
51	9.60 675	9.64 552	0.35 448	9.96 123	9.62 350	9.66 570	0.33 430	9.95 780	9
52	9.60 704	9.64 586	0.35 414	9.96 118	9.62 377	9.66 603	0.33 397	9.95 775	8
53	9.60 732	9.64 620	0.35 380	9.96 112	9.62 405	9.66 636	0.33 364	9.95 769	7
54	9.60 761	9.64 654	0.35 346	9.96 107	9.62 432	9.66 669	0.33 331	9.95 763	6
55	9.60 789	9.64 688	0.35 312	9.96 101	9.62 459	9.66 702	0.33 298	9.95 757	5
56	9.60 818	9.64 722	0.35 278	9.96 095	9.62 486	9.66 735	0.33 265	9.95 751	4
57	9.60 846	9.64 756	0.35 244	9.96 090	9.62 513	9.66 768	0.33 232	9.95 745	3
58	9.60 875	9.64 790	0.35 210	9.96 084	9.62 541	9.66 801	0.33 199	9.95 739	2
59	9.60 903	9.64 824	0.35 176	9.96 079	9.62 568	9.66 834	0.33 166	9.95 733	1
60	9.60 931	9.64 858	0.35 142	9.96 073	9.62 595	9.66 867	0.33 133	9.95 728	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	66°				65°				′

′	25°				26°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.62 595	9.66 867	0.33 133	9.95 728	9.64 184	9.68 818	0.31 182	9.95 366	60
1	9.62 622	9.66 900	0.33 100	9.95 722	9.64 210	9.68 850	0.31 150	9.95 360	59
2	9.62 649	9.66 933	0.33 067	9.95 716	9.64 236	9.68 882	0.31 118	9.95 354	58
3	9.62 676	9.66 966	0.33 034	9.95 710	9.64 262	9.68 914	0.31 086	9.95 348	57
4	9.62 703	9.66 999	0.33 001	9.95 704	9.64 288	9.68 946	0.31 054	9.95 341	56
5	9.62 730	9.67 032	0.32 968	9.95 698	9.64 313	9.68 978	0.31 022	9.95 335	55
6	9.62 757	9.67 065	0.32 935	9.95 692	9.64 339	9.69 010	0.30 990	9.95 329	54
7	9.62 784	9.67 098	0.32 902	9.95 686	9.64 365	9.69 042	0.30 958	9.95 323	53
8	9.62 811	9.67 131	0.32 869	9.95 680	9.64 391	9.69 074	0.30 926	9.95 317	52
9	9.62 838	9.67 163	0.32 837	9.95 674	9.64 417	9.69 106	0.30 894	9.95 310	51
10	9.62 865	9.67 196	0.32 804	9.95 668	9.64 442	9.69 138	0.30 862	9.95 304	50
11	9.62 892	9.67 229	0.32 771	9.95 663	9.64 468	9.69 170	0.30 830	9.95 298	49
12	9.62 918	9.67 262	0.32 738	9.95 657	9.64 494	9.69 202	0.30 798	9.95 292	48
13	9.62 945	9.67 295	0.32 705	9.95 651	9.64 519	9.69 234	0.30 766	9.95 286	47
14	9.62 972	9.67 327	0.32 673	9.95 645	9.64 545	9.69 266	0.30 734	9.95 279	46
15	9.62 999	9.67 360	0.32 640	9.95 639	9.64 571	9.69 298	0.30 702	9.95 273	45
16	9.63 026	9.67 393	0.32 607	9.95 633	9.64 596	9.69 329	0.30 671	9.95 267	44
17	9.63 052	9.67 426	0.32 574	9.95 627	9.64 622	9.69 361	0.30 639	9.95 261	43
18	9.63 079	9.67 458	0.32 542	9.95 621	9.64 647	9.69 393	0.30 607	9.95 254	42
19	9.63 106	9.67 491	0.32 509	9.95 615	9.64 673	9.69 425	0.30 575	9.95 248	41
20	9.63 133	9 67 524	0.32 476	9.95 609	9.64 698	9.69 457	0.30 543	9.95 242	40
21	9.63 159	9.67 556	0.32 444	9.95 603	9.64 724	9.69 488	0.30 512	9.95 236	39
22	9.63 186	9.67 589	0.32 411	9.95 597	9.64 749	9.69 520	0.30 480	9.95 229	38
23	9.63 213	9.67 622	0.32 378	9.95 591	9.64 775	9.69 552	0.30 448	9.95 223	37
24	9.63 239	9.67 654	0.32 346	9.95 585	9.64 800	9.69 584	0.30 416	9.95 217	36
25	9.63 266	9.67 687	0.32 313	9.95 579	9.64 826	9.69 615	0.30 385	9.95 211	35
26	9.63 292	9.67 719	0.32 281	9.95 573	9.64 851	9.69 647	0.30 353	9.95 204	34
27	9.63 319	9.67 752	0.32 248	9.95 567	9.64 877	9.69 679	0.30 321	9.95 198	33
28	9.63 345	9.67 785	0.32 215	9.95 561	9.64 902	9.69 710	0.30 290	9.95 192	32
29	9.63 372	9.67 817	0.32 183	9.95 555	9.64 927	9.69 742	0.30 258	9.95 185	31
30	9.63 398	9.67 850	0.32 150	9.95 549	9.64 953	9.69 774	0.30 226	9.95 179	30
31	9.63 425	9.67 882	0.32 118	9.95 543	9.64 978	9.69 805	0.30 195	9.95 173	29
32	9.63 451	9.67 915	0.32 085	9.95 537	9.65 003	9.69 837	0.30 163	9.95 167	28
33	9.63 478	9.67 947	0.32 053	9.95 531	9.65 029	9.69 868	0.30 132	9.95 160	27
34	9.63 504	9.67 980	0.32 020	9.95 525	9.65 054	9.69 900	0.30 100	9.95 154	26
35	9.63 531	9.68 012	0.31 988	9.95 519	9.65 079	9.69 932	0.30 068	9.95 148	25
36	9.63 557	9.68 044	0.31 956	9.95 513	9.65 104	9.69 963	0.30 037	9.95 141	24
37	9.63 583	9.68 077	0.31 923	9.95 507	9.65 130	9.69 995	0.30 005	9.95 135	23
38	9.63 610	9.68 109	0.31 891	9.95 500	9.65 155	9 70 026	0.29 974	9.95 129	22
39	9.63 636	9.68 142	0.31 858	9.95 494	9.65 180	9.70 058	0 .29 942	9.95 122	21
40	9.63 662	9.68 174	0.31 826	9.95 488	9.65 205	9.70 089	0.29 911	9.95 116	20
41	9.63 689	9.68 206	0.31 794	9.95 482	9.65 230	9.70 121	0.29 879	9.95 110	19
42	9.63 715	9.68 239	0.31 761	9.95 476	9.65 255	9.70 152	0.29 848	9.95 103	18
43	9.63 741	9.68 271	0.31 729	9.95 470	9.65 281	9.70 184	0.29 816	9.95 097	17
44	9.63 767	9.68 303	0.31 697	9.95 464	9.65 306	9.70 215	0.29 785	9.95 090	16
45	9.63 794	9.68 336	0.31 664	9.95 458	9.65 331	9.70 247	0.29 753	9.95 084	15
46	9.63 820	9.68 368	0.31 632	9.95 452	9.65 356	9.70 278	0.29 722	9.95 078	14
47	9.63 846	9.68 400	0.31 600	9.95 446	9.65 381	9.70 309	0.29 691	9.95 071	13
48	9.63 872	9.68 432	0.31 568	9.95 440	9.65 406	9.70 341	0.29 659	9.95 065	12
49	9.63 898	9.68 465	0.31 535	9.95 434	9.65 431	9.70 372	0.29 628	9.95 059	11
50	9.63 924	9.68 497	0.31 503	9.95 427	9.65 456	9.70 404	0.29 596	9.95 052	10
51	9.63 950	9.68 529	0.31 471	9.95 421	9.65 481	9.70 435	0.29 565	9.95 046	9
52	9.63 976	9.68 561	0.31 439	9.95 415	9.65 506	9.70 466	0.29 534	9.95·039	8
53	9.64 002	9.68 593	0.31 407	9.95 409	9.65 531	9.70 498	0.29 502	9.95 033	7
54	9.64 028	9.68 626	0.31 374	9.95 403	9.65 556	9.70 529	0.29 471	9.95 027	6
55	9.64 054	9.68 658	0.31 342	9.95 397	9.65 580	9.70 560	0.29 440	9.95 020	5
56	9.64 080	9.68 690	0.31 310	9.95 391	9.65 605	9.70 592	0.29 408	9 95 014	4
57	9.64 106	9.68 722	0.31 278	9.95 384	9.65 630	9.70 623	0.29 377	9.95 007	3
58	9.64 132	9.68 754	0.31 246	9.95 378	9.65 655	9.70 654	0.29 346	9.95 001	2
59	9.64 158	9.68 786	0.31 214	9.95 372	9.65 680	9.70 685	0.29 315	9.94 995	1
60	9.64 184	9.68 818	0.31 182	9.95 366	9.65 705	9.70 717	0.29 283	9.94 988	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	64°				63°				′

′	27°				28°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.65 705	9.70 717	0.29 283	9.94 988	9.67 161	9.72 567	0.27 433	9.94 593	60
1	9.65 729	9.70 748	0.29 252	9.94 982	9.67 185	9.72 598	0.27 402	9.94 587	59
2	9.65 754	9.70 779	0.29 221	9.94 975	9.67 208	9.72 628	0.27 372	9.94 580	58
3	9.65 779	9.70 810	0.29 190	9.94 969	9.67 232	9.72 659	0.27 341	9.94 573	57
4	9.65 804	9.70 841	0.29 159	9.94 962	9.67 256	9.72 689	0.27 311	9.94 567	56
5	9.65 828	9.70 873	0.29 127	9.94 956	9.67 280	9.72 720	0.27 280	9.94 560	55
6	9.65 853	9.70 904	0.29 096	9.94 949	9.67 303	9.72 750	0.27 250	9.94 553	54
7	9.65 878	9.70 935	0.29 065	9.94 943	9.67 327	9.72 780	0.27 220	9.94 546	53
8	9.65 902	9.70 966	0.29 034	9.94 936	9.67 350	9.72 811	0.27 189	9.94 540	52
9	9.65 927	9.70 997	0.29 003	9.94 930	9.67 374	9.72 841	0.27 159	9.94 533	51
10	9.65 952	9.71 028	0.28 972	9.94 923	9.67 398	9.72 872	0.27 128	9.94 526	50
11	9.65 976	9.71 059	0.28 941	9.94 917	9.67 421	9.72 902	0.27 098	9.94 519	49
12	9.66 001	9.71 090	0.28 910	9.94 911	9.67 445	9.72 932	0.27 068	9.94 513	48
13	9.66 025	9.71 121	0.28 879	9.94 904	9.67 468	9.72 963	0.27 037	9.94 506	47
14	9.66 050	9.71 153	0.28 847	9.94 898	9.67 492	9.72 993	0.27 007	9.94 499	46
15	9.66 075	9.71 184	0.28 816	9.94 891	9.67 515	9.73 023	0.26 977	9.94 492	45
16	9.66 099	9.71 215	0.28 785	9.94 885	9.67 539	9.73 054	0.26 946	9.94 485	44
17	9.66 124	9.71 246	0.28 754	9.94 878	9.67 562	9.73 084	0.26 916	9.94 479	43
18	9.66 148	9.71 277	0.28 723	9.94 871	9.67 586	9.73 114	0.26 886	9.94 472	42
19	9.66 173	9.71 308	0.28 692	9.94 865	9.67 609	9.73 144	0.26 856	9.94 465	41
20	9.66 197	9.71 339	0.28 661	9.94 858	9.67 633	9.73 175	0.26 825	9.94 458	40
21	9.66 221	9.71 370	0.28 630	9.94 852	9.67 656	9.73 205	0.26 795	9.94 451	39
22	9.66 246	9.71 401	0.28 599	9.94 845	9.67 680	9.73 235	0.26 765	9.94 445	38
23	9.66 270	9.71 431	0.28 569	9.94 839	9.67 703	9.73 265-	0.26 735	9.94 438	37
24	9.66 295	9.71 462	0.28 538	9.94 832	9.67 726	9.73 295	0.26 705	9.94 431	36
25	9.66 319	9.71 493	0.28 507	9.94 826	9.67 750	9.73 326	0.26 674	9.94 424	35
26	9.66 343	9.71 524	0.28 476	9.94 819	9.67 773	9.73 356	0.26 644	9.94 417	34
27	9.66 368	9.71 555	0.28 445	9.94 813	9.67 796	9.73 386	0.26 614	9.94 410	33
28	9.66 392	9.71 586	0.28 414	9.94 806	9.67 820	9.73 416	0.26 584	9.94 404	32
29	9.66 416	9.71 617	0.28 383	9.94 799	9.67 843	9.73 446	0.26 554	9.94 397	31
30	9.66 441	9.71 648	0.28 352	9.94 793	9.67 866	9.73 476	0.26 524	9.94 390	30
31	9.66 465	9.71 679	0.28 321	9.94 786	9.67 890	9.73 507	0.26 493	9.94 383	29
32	9.66 489	9.71 709	0.28 291	9.94 780	9.67 913	9.73 537	0.26 463	9.94 376	28
33	9.66 513	9.71 740	0.28 260	9.94 773	9.67 936	9.73 567	0.26 433	9 94 369	27
34	9.66 537	9.71 771	0.28 229	9.94 767	9.67 959	9.73 597	0.26 403	9.94 362	26
35	9.66 562	9.71 802	0.28 198	9.94 760	9.67 982	9.73 627	0.26 373	9.94 355	25
36	9.66 586	9.71 833	0.28 167	9.94 753	9.68 006	9.73 657	0.26 343	9.94 349	24
37	9.66 610	9.71 863	0.28 137	9.94 747	9.68 029	9.73 687	0.26 313	9.94 342	23
38	9.66 634	9.71 894	0.28 106	9.94 740	9.68 052	9.73 717	0.26 283	9.94 335	22
39	9.66 658	9.71 925	0.28 075	9.94 734	9.68 075	9.73 747	0.26 253	9.94 328	21
40	9.66 682	9.71 955	0.28 045	9.94 727	9.68 098	9.73 777	0.26 223	9.94 321	20
41	9.66 706	9.71 986	0.28 014	9.94 720	9.68 121	9.73 807	0.26 193	9.94 314	19
42	9.66 731	9.72 017	0.27 983	9.94 714	9.68 144	9.73 837	0.26 163	9.94 307	18
43	9.66 755	9.72 048	0.27 952	9.94 707	9.68 167	9.73 867	0.26 133	9.94 300	17
44	9.66 779	9.72 078	0.27 922	9.94 700	9.68 190	9.73 897	0.26 103	9.94 293	16
45	9.66 803	9.72 109	0.27 891	9.94 694	9.68 213	9.73 927	0.26 073	9.94 286	15
46	9.66 827	9.72 140	0.27 860	9.94 687	9.68 237	9.73 957	0.26 043	9.94 279	14
47	9.66 851	9.72 170	0.27 830	9.94 680	9.68 260	9.73 987	0.26 013	9.94 273	13
48	9.66 875	9.72 201	0.27 799	9.94 674	9.68 283	9.74 017	0.25 983	9.94 266	12
49	9.66 899	9.72 231	0.27 769	9.94 667	9.68 305	9.74 047	0.25 953	9.94 259	11
50	9.66 922	9.72 262	0.27 738	9.94 660	9.68 328	9.74 077	0.25 923	9.94 252	10
51	9.66 946	9.72 293	0.27 707	9.94 654	9.68 351	9.74 107	0.25 893	9.94 245	9
52	9.66 970	9.72 323	0.27 677	9.94 647	9.68 374	9.74 137	0.25 863	9.94 238	8
53	9.66 994	9.72 354	0.27 646	9.94 640	9.68 397	9.74 166	0.25 834	9.94 231	7
54	9.67 018	9.72 384	0.27 616	9.94 634	9.68 420	9.74 196	0.25 804	9.94 224	6
55	9.67 042	9.72 415-	0.27 585	9.94 627	9.68 443	9.74 226	0.25 774	9.94 217	5
56	9.67 066	9.72 445	0.27 555	9.94 620	9.68 466	9.74 256	0.25 744	9.94 210	4
57	9.67 090	9.72 476	0.27 524	9.94 614	9.68 489	9.74 286	0.25 714	9.94 203	3
58	9.67 113	9.72 506	0.27 494	9.94 607	9.68 512	9.74 316	0.25 684	9.94 196	2
59	9.67 137	9.72 537	0.27 463	9.94 600	9.68 534	9.74 345	0.25 655	9.94 189	1
60	9.67 161	9.72 567	0.27 433	9.94 593	9.68 557	9.74 375	0 25 625	9.94 182	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	62°				61°				′

′	29°				30°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.68 557	9.74 375	0.25 625	9.94 182	9.69 897	9.76 144	0.23 856	9.93 753	**60**
1	9.68 580	9.74 405	0.25 595	9.94 175	9.69 919	9.76 173	0.23 827	9.93 746	59
2	9.68 603	9.74 435	0.25 565	9.94 168	9.69 941	9.76 202	0.23 798	9.93 738	58
3	9.68 625	9.74 465	0.25 535	9.94 161	9.69 963	9.76 231	0.23 769	9.93 731	57
4	9.68 648	9.74 494	0.25 506	9.94 154	9.69 984	9.76 261	0.23 739	9.93 724	56
5	9.68 671	9.74 524	0.25 476	9.94 147	9.70 006	9.76 290	0.23 710	9.93 717	55
6	9.68 694	9.74 554	0.25 446	9.94 140	9.70 028	9.76 319	0.23 681	9.93 709	54
7	9.68 716	9.74 583	0.25 417	9.94 133	9.70 050	9.76 348	0.23 652	9.93 702	53
8	9.68 739	9.74 613	0.25 387	9.94 126	9.70 072	9.76 377	0.23 623	9.93 695	52
9	9.68 762	9.74 643	0.25 357	9.94 119	9.70 093	9.76 406	0.23 594	9.93 687	51
10	9.68 784	9.74 673	0.25 327	9.94 112	9.70 115	9.76 435	0.23 565	9.93 680	**50**
11	9.68 807	9.74 702	0.25 298	9.94 105	9.70 137	9.76 464	0.23 536	9.93 673	49
12	9.68 829	9.74 732	0.25 268	9.94 098	9.70 159	9.76 493	0.23 507	9.93 665	48
13	9.68 852	9.74 762	0.25 238	9.94 090	9.70 180	9.76 522	0.23 478	9.93 658	47
14	9.68 875	9.74 791	0.25 209	9.94 083	9.70 202	9.76 551	0.23 449	9.93 650	46
15	9.68 897	9.74 821	0.25 179	9.94 076	9.70 224	9.76 580	0.23 420	9.93 643	45
16	9.68 920	9.74 851	0.25 149	9.94 069	9.70 245	9.76 609	0.23 391	9.93 636	44
17	9.68 942	9.74 880	0.25 120	9.94 062	9.70 267	9.76 639	0.23 361	9.93 628	43
18	9.68 965	9.74 910	0.25 090	9.94 055	9.70 288	9.76 668	0.23 332	9.93 621	42
19	9.68 987	9.74 939	0.25 061	9.94 048	9.70 310	9.76 697	0.23 303	9.93 614	41
20	9.69 010	9.74 969	0.25 031	9.94 041	9.70 332	9.76 725	0.23 275	9.93 606	**40**
21	9.69 032	9.74 998	0.25 002	9.94 034	9.70 353	9.76 754	0.23 246	9.93 599	39
22	9.69 055	9.75 028	0.24 972	9.94 027	9.70 375	9.76 783	0.23 217	9.93 591	38
23	9.69 077	9.75 058	0.24 942	9.94 020	9.70 396	9.76 812	0.23 188	9.93 584	37
24	9.69 100	9.75 087	0.24 913	9.94 012	9.70 418	9.76 841	0.23 159	9.93 577	36
25	9.69 122	9.75 117	0.24 883	9.94 005	9.70 439	9.76 870	0.23 130	9.93 569	35
26	9.69 144	9.75 146	0.24 854	9.93 998	9.70 461	9.76 899	0.23 101	9.93 562	34
27	9.69 167	9.75 176	0.24 824	9.93 991	9.70 482	9.76 928	0.23 072	9.93 554	33
28	9.69 189	9.75 205	0.24 795	9.93 984	9.70 504	9.76 957	0.23 043	9.93 547	32
29	9.69 212	9.75 235	0.24 765	9.93 977	9.70 525	9.76 986	0.23 014	9.93 539	31
30	9.69 234	9.75 264	0.24 736	9.93 970	9.70 547	9.77 015	0.22 985	9.93 532	**30**
31	9.69 256	9.75 294	0.24 706	9.93 963	9.70 568	9.77 044	0.22 956	9.93 525	29
32	9.69 279	9.75 323	0.24 677	9.93 955	9.70 590	9.77 073	0.22 927	9.93 517	28
33	9.69 301	9.75 353	0.24 647	9.93 948	9.70 611	9.77 101	0.22 899	9.93 510	27
34	9.69 323	9.75 382	0.24 618	9.93 941	9.70 633	9.77 130	0.22 870	9.93 502	26
35	9.69 345	9.75 411	0.24 589	9.93 934	9.70 654	9.77 159	0.22 841	9.93 495	25
36	9.69 368	9.75 441	0.24 559	9.93 927	9.70 675	9.77 188	0.22 812	9.93 487	24
37	9.69 390	9.75 470	0.24 530	9.93 920	9.70 697	9.77 217	0.22 783	9.93 480	23
38	9.69 412	9.75 500	0.24 500	9.93 912	9.70 718	9.77 246	0.22 754	9.93 472	22
39	9.69 434	9.75 529	0.24 471	9.93 905	9.70 739	9.77 274	0.22 726	9.93 465	21
40	9.69 456	9.75 558	0.24 442	9.93 898	9.70 761	9.77 303	0.22 697	9.93 457	**20**
41	9.69 479	9.75 588	0.24 412	9.93 891	9.70 782	9.77 332	0.22 668	9.93 450	19
42	9.69 501	9.75 617	0.24 383	9.93 884	9.70 803	9.77 361	0.22 639	9.93 442	18
43	9.69 523	9.75 647	0.24 353	9.93 876	9.70 824	9.77 390	0.22 610	9.93 435	17
44	9.69 545	9.75 676	0.24 324	9.93 869	9.70 846	9.77 418	0.22 582	9.93 427	16
45	9.69 567	9.75 705	0.24 295	9.93 862	9.70 867	9.77 447	0.22 553	9.93 420	15
46	9.69 589	9.75 735	0.24 265	9.93 855	9.70 888	9.77 476	0.22 524	9.93 412	14
47	9.69 611	9.75 764	0.24 236	9.93 847	9.70 909	9.77 505	0.22 495	9.93 405	13
48	9.69 633	9.75 793	0.24 207	9.93 840	9.70 931	9.77 533	0.22 467	9.93 397	12
49	9.69 655	9.75 822	0.24 178	9.93 833	9.70 952	9.77 562	0.22 438	9.93 390	11
50	9.69 677	9.75 852	0.24 148	9.93 826	9.70 973	9.77 591	0.22 409	9.93 382	**10**
51	9.69 699	9.75 881	0.24 119	9.93 819	9.70 994	9.77 619	0.22 381	9.93 375	9
52	9.69 721	9.75 910	0.24 090	9.93 811	9.71 015	9.77 648	0.22 352	9.93 367	8
53	9.69 743	9.75 939	0.24 061	9.93 804	9.71 036	9.77 677	0.22 323	9.93 360	7
54	9.69 765	9.75 969	0.24 031	9.93 797	9.71 058	9.77 706	0.22 294	9.93 352	6
55	9.69 787	9.75 998	0.24 002	9.93 789	9.71 079	9.77 734	0.22 266	9.93 344	5
56	9.69 809	9.76 027	0.23 973	9.93 782	9.71 100	9.77 763	0.22 237	9.93 337	4
57	9.69 831	9.76 056	0.23 944	9.93 775	9.71 121	9.77 791	0.22 209	9.93 329	3
58	9.69 853	9.76 086	0.23 914	9.93 768	9.71 142	9.77 820	0.22 180	9.93 322	2
59	9.69 875	9.76 115	0.23 885	9.93 760	9.71 163	9.77 849	0.22 151	9.93 314	1
60	9.69 897	9.76 144	0.23 856	9.93 753	9.71 184	9.77 877	0.22 123	9.93 307	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	60°				59°				′

′	31°				32°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.71 184	9.77 877	0.22 123	9.93 307	9.72 421	9.79 579	0.20 421	9.92 842	60
1	9.71 205	9.77 906	0.22 094	9.93 299	9.72 441	9.79 607	0.20 393	9.92 834	59
2	9.71 226	9.77 935	0.22 065	9.93 291	9.72 461	9.79 635	0.20 365	9.92 826	58
3	9.71 247	9.77 963	0.22 037	9.93 284	9.72 482	9.79 663	0.20 337	9.92 818	57
4	9.71 268	9.77 992	0.22 008	9.93 276	9.72 502	9.79 691	0.20 309	9.92 810	56
5	9.71 289	9.78 020	0.21 980	9.93 269	9.72 522	9.79 719	0.20 281	9.92 803	55
6	9.71 310	9.78 049	0.21 951	9.93 261	9.72 542	9.79 747	0.20 253	9.92 795	54
7	9.71 331	9.78 077	0.21 923	9.93 253	9.72 562	9.79 776	0.20 224	9.92 787	53
8	9.71 352	9.78 106	0.21 894	9.93 246	9.72 582	9.79 804	0.20 196	9.92 779	52
9	9.71 373	9.78 135	0.21 865	9.93 238	9.72 602	9.79 832	0.20 168	9.92 771	51
10	9.71 393	9.78 163	0.21 837	9.93 230	9.72 622	9.79 860	0.20 140	9.92 763	50
11	9.71 414	9.78 192	0.21 808	9.93 223	9.72 643	9.79 888	0.20 112	9.92 755	49
12	9.71 435	9.78 220	0.21 780	9.93 215	9.72 663	9.79 916	0.20 084	9.92 747	48
13	9.71 456	9.78 249	0.21 751	9.93 207	9.72 683	9.79 944	0.20 056	9.92 739	47
14	9.71 477	9.78 277	0.21 723	9.93 200	9.72 703	9.79 972	0.20 028	9.92 731	46
15	9.71 498	9.78 306	0.21 694	9.93 192	9.72 723	9.80 000	0.20 000	9.92 723	45
16	9.71 519	9.78 334	0.21 666	9.93 184	9.72 743	9.80 028	0.19 972	9.92 715	44
17	9.71 539	9.78 363	0.21 637	9.93 177	9.72 763	9.80 056	0.19 944	9.92 707	43
18	9.71 560	9.78 391	0.21 609	9.93 169	9.72 783	9.80 084	0.19 916	9.92 699	42
19	9.71 581	9.78 419	0.21 581	9.93 161	9.72 803	9.80 112	0.19 888	9.92 691	41
20	9.71 602	9.78 448	0.21 552	9.93 154	9.72 823	9.80 140	0.19 860	9.92 683	40
21	9.71 622	9.78 476	0.21 524	9.93 146	9.72 843	9.80 168	0.19 832	9.92 675	39
22	9.71 643	9.78 505	0.21 495	9.93 138	9.72 863	9.80 195	0.19 805	9.92 667	38
23	9.71 664	9.78 533	0.21 467	9.93 131	9.72 883	9.80 223	0.19 777	9.92 659	37
24	9.71 685	9.78 562	0.21 438	9.93 123	9.72 902	9.80 251	0.19 749	9.92 651	36
25	9.71 705	9.78 590	0.21 410	9.93 115	9.72 922	9.80 279	0.19 721	9.92 643	35
26	9.71 726	9.78 618	0.21 382	9.93 108	9.72 942	9.80 307	0.19 693	9.92 635	34
27	9.71 747	9.78 647	0.21 353	9.93 100	9.72 962	9.80 335	0.19 665	9.92 627	33
28	9.71 767	9.78 675	0.21 325	9.93 092	9.72 982	9.80 363	0.19 637	9.92 619	32
29	9.71 788	9.78 704	0.21 296	9.93 084	9.73 002	9.80 391	0.19 609	9.92 611	31
30	9.71 809	9.78 732	0.21 268	9.93 077	9.73 022	9.80 419	0.19 581	9.92 603	30
31	9.71 829	9.78 760	0.21 240	9.93 069	9.73 041	9.80 447	0.19 553	9.92 595	29
32	9.71 850	9.78 789	0.21 211	9.93 061	9.73 061	9.80 474	0.19 526	9.92 587	28
33	9.71 870	9.78 817	0.21 183	9.93 053	9.73 081	9.80 502	0.19 498	9.92 579	27
34	9.71 891	9.78 845	0.21 155	9.93 046	9.73 101	9.80 530	0.19 470	9.92 571	26
35	9.71 911	9.78 874	0.21 126	9.93 038	9.73 121	9.80 558	0.19 442	9.92 563	25
36	9.71 932	9.78 902	0.21 098	9.93 030	9.73 140	9.80 586	0.19 414	9.92 555	24
37	9.71 952	9.78 930	0.21 070	9.93 022	9.73 160	9.80 614	0.19 386	9.92 546	23
38	9.71 973	9.78 959	0.21 041	9.93 014	9.73 180	9.80 642	0.19 358	9.92 538	22
39	9.71 994	9.78 987	0.21 013	9.93 007	9.73 200	9.80 669	0.19 331	9.92 530	21
40	9.72 014	9.79 015	0.20 985	9.92 999	9.73 219	9.80 697	0.19 303	9.92 522	20
41	9.72 034	9.79 043	0.20 957	9.92 991	9.73 239	9.80 725	0.19 275	9.92 514	19
42	9.72 055	9.79 072	0.20 928	9.92 983	9.73 259	9.80 753	0.19 247	9.92 506	18
43	9.72 075	9.79 100	0.20 900	9.92 976	9.73 278	9.80 781	0.19 219	9.92 498	17
44	9.72 096	9.79 128	0.20 872	9.92 968	9.73 298	9.80 808	0.19 192	9.92 490	16
45	9.72 116	9.79 156	0.20 844	9.92 960	9.73 318	9.80 836	0.19 164	9.92 482	15
46	9.72 137	9.79 185	0.20 815	9.92 952	9.73 337	9.80 864	0.19 136	9.92 473	14
47	9.72 157	9.79 213	0.20 787	9.92 944	9.73 357	9.80 892	0.19 108	9.92 465	13
48	9.72 177	9.79 241	0.20 759	9.92 936	9.73 377	9.80 919	0.19 081	9.92 457	12
49	9.72 198	9.79 269	0.20 731	9.92 929	9.73 396	9.80 947	0.19 053	9.92 449	11
50	9.72 218	9.79 297	0.20 703	9.92 921	9.73 416	9.80 975	0.19 025	9.92 441	10
51	9.72 238	9.79 326	0.20 674	9.92 913	9.73 435	9.81 003	0.18 997	9.92 433	9
52	9.72 259	9.79 354	0.20 646	9.92 905	9.73 455	9.81 030	0.18 970	9.92 425	8
53	9.72 279	9.79 382	0.20 618	9.92 897	9.73 474	9.81 058	0.18 942	9.92 416	7
54	9.72 299	9.79 410	0.20 590	9.92 889	9.73 494	9.81 086	0.18 914	9.92 408	6
55	9.72 320	9.79 438	0.20 562	9.92 881	9.73 513	9.81 113	0.18 887	9.92 400	5
56	9.72 340	9.79 466	0.20 534	9.92 874	9.73 533	9.81 141	0.18 859	9.92 392	4
57	9.72 360	9.79 495	0.20 505	9.92 866	9.73 552	9.81 169	0.18 831	9.92 384	3
58	9.72 381	9.79 523	0.20 477	9.92 858	9.73 572	9.81 196	0.18 804	9.92 376	2
59	9.72 401	9.79 551	0.20 449	9.92 850	9.73 591	9.81 224	0.18 776	9.92 367	1
60	9.72 421	9.79 579	0.20 421	9.92 842	9.73 611	9.81 252	0.18 748	9.92 359	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	58°				57°				′

′	33°				34°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.73 611	9.81 252	0.18 748	9.92 359	9.74 756	9.82 899	0.17 101	9.91 857	60
1	9.73 630	9.81 279	0.18 721	9.92 351	9.74 775	9.82 926	0.17 074	9.91 849	59
2	9.73 650	9.81 307	0.18 693	9.92 343	9.74 794	9.82 953	0.17 047	9.91 840	58
3	9.73 669	9.81 335	0.18 665	9.92 335	9.74 812	9.82 980	0.17 020	9.91 832	57
4	9.73 689	9.81 362	0.18 638	9.92 326	9.74 831	9.83 008	0.16 992	9.91 823	56
5	9.73 708	9.81 390	0.18 610	9.92 318	9.74 850	9.83 035	0.16 965	9.91 815	55
6	9.73 727	9.81 418	0.18 582	9.92 310	9.74 868	9.83 062	0.16 938	9.91 806	54
7	9.73 747	9.81 445	0.18 555	9.92 302	9.74 887	9.83 089	0.16 911	9.91 798	53
8	9.73 766	9.81 473	0.18 527	9.92 293	9.74 906	9.83 117	0.16 883	9.91 789	52
9	9.73 785	9.81 500	0.18 500	9.92 285	9.74 924	9.83 144	0.16 856	9.91 781	51
10	9.73 805	9.81 528	0.18 472	9.92 277	9.74 943	9.83 171	0.16 829	9.91 772	50
11	9.73 824	9.81 556	0.18 444	9.92 269	9.74 961	9.83 198	0.16 802	9.91 763	49
12	9.73 843	9.81 583	0.18 417	9.92 260	9.74 980	9.83 225	0.16 775	9.91 755	48
13	9.73 863	9.81 611	0.18 389	9.92 252	9.74 999	9.83 252	0.16 748	9.91 746	47
14	9.73 882	9.81 638	0.18 362	9.92 244	9.75 017	9.83 280	0.16 720	9.91 738	46
15	9.73 901	9.81 666	0.18 334	9.92 235	9.75 036	9.83 307	0.16 693	9.91 729	45
16	9.73 921	9.81 693	0.18 307	9.92 227	9.75 054	9.83 334	0.16 666	9.91 720	44
17	9.73 940	9.81 721	0.18 279	9.92 219	9.75 073	9.83 361	0.16 639	9.91 712	43
18	9.73 959	9.81 748	0.18 252	9.92 211	9.75 091	9.83 388	0.16 612	9.91 703	42
19	9.73 978	9.81 776	0.18 224	9.92 202	9.75 110	9.83 415	0.16 585	9.91 695	41
20	9.73 997	9.81 803	0.18 197	9.92 194	9.75 128	9.83 442	0.16 558	9.91 686	40
21	9.74 017	9.81 831	0.18 169	9.92 186	9.75 147	9.83 470	0.16 530	9.91 677	39
22	9.74 036	9.81 858	0.18 142	9.92 177	9.75 165	9.83 497	0.16 503	9.91 669	38
23	9.74 055	9.81 886	0.18 114	9.92 169	9.75 184	9.83 524	0.16 476	9.91 660	37
24	9.74 074	9.81 913	0.18 087	9.92 161	9.75 202	9.83 551	0.16 449	9.91 651	36
25	9.74 093	9.81 941	0.18 059	9.92 152	9.75 221	9.83 578	0.16 422	9.91 643	35
26	9.74 113	9.81 968	0.18 032	9.92 144	9.75 239	9.83 605	0.16 395	9.91 634	34
27	9.74 132	9.81 996	0.18 004	9.92 136	9.75 258	9.83 632	0.16 368	9.91 625	33
28	9.74 151	9.82 023	0.17 977	9.92 127	9.75 276	9.83 659	0.16 341	9.91 617	32
29	9.74 170	9.82 051	0.17 949	9.92 119	9.75 294	9.83 686	0.16 314	9.91 608	31
30	9.74 189	9.82 078	0.17 922	9.92 111	9.75 313	9.83 713	0.16 287	9.91 599	30
31	9.74 208	9.82 106	0.17 894	9.92 102	9.75 331	9.83 740	0.16 260	9.91 591	29
32	9.74 227	9.82 133	0.17 867	9.92 094	9.75 350	9.83 768	0.16 232	9.91 582	28
33	9.74 246	9.82 161	0.17 839	9.92 086	9.75 368	9.83 795	0.16 205	9.91 573	27
34	9.74 265	9.82 188	0.17 812	9.92 077	9.75 386	9.83 822	0.16 178	9.91 565	26
35	9.74 284	9.82 215	0.17 785	9.92 069	9.75 405	9.83 849	0.16 151	9.91 556	25
36	9.74 303	9.82 243	0.17 757	9.92 060	9.75 423	9.83 876	0.16 124	9.91 547	24
37	9.74 322	9.82 270	0.17 730	9.92 052	9.75 441	9.83 903	0.16 097	9.91 538	23
38	9.74 341	9.82 298	0.17 702	9.92 044	9.75 459	9.83 930	0.16 070	9.91 530	22
39	9.74 360	9.82 325	0.17 675	9.92 035	9.75 478	9.83 957	0.16 043	9.91 521	21
40	9.74 379	9.82 352	0.17 648	9.92 027	9.75 496	9.83 984	0.16 016	9.91 512	20
41	9.74 398	9.82 380	0.17 620	9.92 018	9.75 514	9.84 011	0.15 989	9.91 504	19
42	9.74 417	9.82 407	0.17 593	9.92 010	9.75 533	9.84 038	0.15 962	9.91 495	18
43	9.74 436	9.82 435	0.17 565	9.92 002	9.75 551	9.84 065	0.15 935	9.91 486	17
44	9.74 455	9.82 462	0.17 538	9.91 993	9.75 569	9.84 092	0.15 908	9.91 477	16
45	9.74 474	9.82 489	0.17 511	9.91 985	9.75 587	9.84 119	0.15 881	9.91 469	15
46	9.74 493	9.82 517	0.17 483	9.91 976	9.75 605	9.84 146	0.15 854	9.91 460	14
47	9.74 512	9.82 544	0.17 456	9.91 968	9.75 624	9.84 173	0.15 827	9.91 451	13
48	9.74 531	9.82 571	0.17 429	9.91 959	9.75 642	9.84 200	0.15 800	9.91 442	12
49	9.74 549	9.82 599	0.17 401	9.91 951	9.75 660	9.84 227	0.15 773	9.91 433	11
50	9.74 568	9.82 626	0.17 374	9.91 942	9.75 678	9.84 254	0.15 746	9.91 425	10
51	9.74 587	9.82 653	0.17 347	9.91 934	9.75 696	9.84 280	0.15 720	9.91 416	9
52	9.74 606	9.82 681	0.17 319	9.91 925	9.75 714	9.84 307	0.15 693	9.91 407	8
53	9.74 625	9.82 708	0.17 292	9.91 917	9.75 733	9.84 334	0.15 666	9.91 398	7
54	9.74 644	9.82 735	0.17 265	9.91 908	9.75 751	9.84 361	0.15 639	9.91 389	6
55	9.74 662	9.82 762	0.17 238	9.91 900	9.75 769	9.84 388	0.15 612	9.91 381	5
56	9.74 681	9.82 790	0.17 210	9.91 891	9.75 787	9.84 415	0.15 585	9.91 372	4
57	9.74 700	9.82 817	0.17 183	9.91 883	9.75 805	9.84 442	0.15 558	9.91 363	3
58	9.74 719	9.82 844	0.17 156	9.91 874	9.75 823	9.84 469	0.15 531	9.91 354	2
59	9.74 737	9.82 871	0.17 129	9.91 866	9.75 841	9.84 496	0.15 504	9.91 345	1
60	9.74 756	9.82 899	0.17 101	9.91 857	9.75 859	9.84 523	0.15 477	9.91 336	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
	56°				55°				′

′	35°				36°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.75 859	9.84 523	0.15 477	9.91 336	9.76 922	9.86 126	0.13 874	9.90 796	60
1	9.75 877	9.84 550	0.15 450	9.91 328	9.76 939	9.86 153	0.13 847	9.90 787	59
2	9.75 895	9.84 576	0.15 424	9.91 319	9.76 957	9.86 179	0.13 821	9.90 777	58
3	9.75 913	9.84 603	0.15 397	9.91 310	9.76 974	9.86 206	0.13 794	9.90 768	57
4	9.75 931	9.84 630	0.15 370	9.91 301	9.76 991	9.86 232	0.13 768	9.90 759	56
5	9.75 949	9.84 657	0.15 343	9.91 292	9.77 009	9.86 259	0.13 741	9.90 750	55
6	9.75 967	9.84 684	0.15 316	9.91 283	9.77 026	9.86 285	0.13 715	9.90 741	54
7	9.75 985	9.84 711	0.15 289	9.91 274	9.77 043	9.86 312	0.13 688	9.90 731	53
8	9.76 003	9.84 738	0.15 262	9.91 266	9.77 061	9.86 338	0.13 662	9.90 722	52
9	9.76 021	9.84 764	0.15 236	9.91 257	9.77 078	9.86 365	0.13 635	9.90 713	51
10	9.76 039	9.84 791	0.15 209	9.91 248	9.77 095	9.86 392	0.13 608	9.90 704	50
11	9.76 057	9.84 818	0.15 182	9.91 239	9.77 112	9.86 418	0.13 582	9.90 694	49
12	9.76 075	9.84 845	0.15 155	9.91 230	9.77 130	9.86 445	0.13 555	9.90 685	48
13	9.76 093	9.84 872	0.15 128	9.91 221	9.77 147	9.86 471	0.13 529	9.90 676	47
14	9.76 111	9.84 899	0.15 101	9.91 212	9.77 164	9.86 498	0.13 502	9.90 667	46
15	9.76 129	9.84 925	0.15 075	9.91 203	9.77 181	9.86 524	0.13 476	9.90 657	45
16	9.76 146	9.84 952	0.15 048	9.91 194	9.77 199	9.86 551	0.13 449	9.90 648	44
17	9.76 164	9.84 979	0.15 021	9.91 185	9.77 216	9.86 577	0.13 423	9.90 639	43
18	9.76 182	9.85 006	0.14 994	9.91 176	9.77 233	9.86 603	0.13 397	9.90 630	42
19	9.76 200	9.85 033	0.14 967	9.91 167	9.77 250	9.86 630	0.13 370	9.90 620	41
20	9.76 218	9.85 059	0.14 941	9.91 158	9.77 268	9.86 656	0.13 344	9.90 611	40
21	9.76 236	9.85 086	0.14 914	9.91 149	9.77 285	9.86 683	0.13 317	9.90 602	39
22	9.76 253	9.85 113	0.14 887	9.91 141	9.77 302	9.86 709	0.13 291	9.90 592	38
23	9.76 271	9.85 140	0.14 860	9.91 132	9.77 319	9.86 736	0.13 264	9.90 583	37
24	9.76 289	9.85 166	0.14 834	9.91 123	9.77 336	9.86 762	0.13 238	9.90 574	36
25	9.76 307	9.85 193	0.14 807	9.91 114	9.77 353	9.86 789	0.13 211	9.90 565	35
26	9.76 324	9.85 220	0.14 780	9.91 105	9.77 370	9.86 815	0.13 185	9.90 555	34
27	9.76 342	9.85 247	0.14 753	9.91 096	9.77 387	9.86 842	0.13 158	9.90 546	33
28	9.76 360	9.85 273	0.14 727	9.91 087	9.77 405	9.86 868	0.13 132	9.90 537	32
29	9.76 378	9.85 300	0.14 700	9.91 078	9.77 422	9.86 894	0.13 106	9.90 527	31
30	9.76 395	9.85 327	0.14 673	9.91 069	9.77 439	9.86 921	0.13 079	9.90 518	30
31	9.76 413	9.85 354	0.14 646	9.91 060	9.77 456	9.86 947	0.13 053	9.90 509	29
32	9.76 431	9.85 380	0.14 620	9.91 051	9.77 473	9.86 974	0.13 026	9.90 499	28
33	9.76 448	9.85 407	0.14 593	9.91 042	9.77 490	9.87 000	0.13 000	9.90 490	27
34	9.76 466	9.85 434	0.14 566	9.91 033	9.77 507	9.87 027	0.12 973	9.90 480	26
35	9.76 484	9.85 460	0.14 540	9.91 023	9.77 524	9.87 053	0.12 947	9.90 471	25
36	9.76 501	9.85 487	0.14 513	9.91 014	9.77 541	9.87 079	0.12 921	9.90 462	24
37	9.76 519	9.85 514	0.14 486	9.91 005	9.77 558	9.87 106	0.12 894	9.90 452	23
38	9.76 537	9.85 540	0.14 460	9.90 996	9.77 575	9.87 132	0.12 868	9.90 443	22
39	9.76 554	9.85 567	0.14 433	9.90 987	9.77 592	9.87 158	0.12 842	9.90 434	21
40	9.76 572	9.85 594	0.14 406	9.90 978	9.77 609	9.87 185	0.12 815	9.90 424	20
41	9.76 590	9.85 620	0.14 380	9.90 969	9.77 626	9.87 211	0.12 789	9.90 415	19
42	9.76 607	9.85 647	0.14 353	9.90 960	9.77 643	9.87 238	0.12 762	9.90 405	18
43	9.76 625	9.85 674	0.14 326	9.90 951	9.77 660	9.87 264	0.12 736	9.90 396	17
44	9.76 642	9.85 700	0.14 300	9.90 942	9.77 677	9.87 290	0.12 710	9.90 386	16
45	9.76 660	9.85 727	0.14 273	9.90 933	9.77 694	9.87 317	0.12 683	9.90 377	15
46	9.76 677	9.85 754	0.14 246	9.90 924	9.77 711	9.87 343	0.12 657	9.90 368	14
47	9.76 695	9.85 780	0.14 220	9.90 915	9.77 728	9.87 369	0.12 631	9.90 358	13
48	9.76 712	9.85 807	0.14 193	9.90 906	9.77 744	9.87 396	0.12 604	9.90 349	12
49	9.76 730	9.85 834	0.14 166	9.90 896	9.77 761	9.87 422	0.12 578	9.90 339	11
50	9.76 747	9.85 860	0.14 140	9.90 887	9.77 778	9.87 448	0.12 552	9.90 330	10
51	9.76 765	9.85 887	0.14 113	9.90 878	9.77 795	9.87 475	0.12 525	9.90 320	9
52	9.76 782	9.85 913	0.14 087	9.90 869	9.77 812	9.87 501	0.12 499	9.90 311	8
53	9.76 800	9.85 940	0.14 060	9.90 860	9.77 829	9.87 527	0.12 473	9.90 301	7
54	9.76 817	9.85 967	0.14 033	9.90 851	9.77 846	9.87 554	0.12 446	9.90 292	6
55	9.76 835	9.85 993	0.14 007	9.90 842	9.77 862	9.87 580	0.12 420	9.90 282	5
56	9.76 852	9.86 020	0.13 980	9.90 832	9.77 879	9.87 606	0.12 394	9.90 273	4
57	9.76 870	9.86 046	0.13 954	9.90 823	9.77 896	9.87 633	0.12 367	9.90 263	3
58	9.76 887	9.86 073	0.13 927	9.90 814	9.77 913	9.87 659	0.12 341	9.90 254	2
59	9.76 904	9.86 100	0.13 900	9.90 805	9.77 930	9.87 685	0.12 315	9.90 244	1
60	9.76 922	9.86 126	0.13 874	9.90 796	9.77 946	9.87 711	0.12 289	9.90 235	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	54°				53°				′

′	37°				38°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.77 946	9.87 711	0.12 289	9.90 235	9.78 934	9.89 281	0.10 719	9.89 653	**60**
1	9.77 963	9.87 738	0.12 262	9.90 225	9.78 950	9.89 307	0.10 693	9.89 643	59
2	9.77 980	9.87 764	0.12 236	9.90 216	9.78 967	9.89 333	0.10 667	9.89 633	58
3	9.77 997	9.87 790	0.12 210	9.90 206	9.78 983	9.89 359	0.10 641	9.89 624	57
4	9.78 013	9.87 817	0.12 183	9.90 197	9.78 999	9.89 385	0.10 615	9.89 614	56
5	9.78 030	9.87 843	0.12 157	9.90 187	9.79 015	9.89 411	0.10 589	9.89 604	55
6	9.78 047	9.87 869	0.12 131	9.90 178	9.79 031	9.89 437	0.10 563	9.89 594	54
7	9.78 063	9.87 895	0.12 105	9.90 168	9.79 047	9.89 463	0.10 537	9.89 584	53
8	9.78 080	9.87 922	0.12 078	9.90 159	9.79 063	9.89 489	0.10 511	9.89 574	52
9	9.78 097	9.87 948	0.12 052	9.90 149	9.79 079	9.89 515	0.10 485	9.89 564	51
10	9.78 113	9.87 974	0.12 026	9.90 139	9.79 095	9.89 541	0.10 459	9.89 554	**50**
11	9.78 130	9.88 000	0.12 000	9.90 130	9.79 111	9.89 567	0.10 433	9.89 544	49
12	9.78 147	9.88 027	0.11 973	9.90 120	9.79 128	9.89 593	0.10 407	9.89 534	48
13	9.78 163	9.88 053	0.11 947	9.90 111	9.79 144	9.89 619	0.10 381	9.89 524	47
14	9.78 180	9.88 079	0.11 921	9.90 101	9.79 160	9.89 645	0.10 355	9.89 514	46
15	9.78 197	9.88 105	0.11 895	9.90 091	9.79 176	9.89 671	0.10 329	9.89 504	45
16	9.78 213	9.88 131	0.11 869	9.90 082	9.79 192	9.89 697	0.10 303	9.89 495	44
17	9.78 230	9.88 158	0.11 842	9.90 072	9.79 208	9.89 723	0.10 277	9.89 485	43
18	9.78 246	9.88 184	0.11 816	9.90 063	9.79 224	9.89 749	0.10 251	9.89 475	42
19	9.78 263	9.88 210	0.11 790	9.90 053	9.79 240	9.89 775	0.10 225	9.89 465	41
20	9.78 280	9.88 236	0.11 764	9.90 043	9.79 256	9.89 801	0.10 199	9.89 455	**40**
21	9.78 296	9.88 262	0.11 738	9.90 034	9.79 272	9.89 827	0.10 173	9.89 445	39
22	9.78 313	9.88 289	0.11 711	9.90 024	9.79 288	9.89 853	0.10 147	9.89 435	38
23	9.78 329	9.88 315	0.11 685	9.90 014	9.79 304	9.89 879	0.10 121	9.89 425	37
24	9.78 346	9.88 341	0.11 659	9.90 005	9.79 319	9.89 905	0.10 095	9.89 415	36
25	9.78 362	9.88 367	0.11 633	9.89 995	9.79 335	9.89 931	0.10 069	9.89 405	35
26	9.78 379	9.88 393	0.11 607	9.89 985	9.79 351	9.89 957	0.10 043	9.89 395	34
27	9.78 395	9.88 420	0.11 580	9.89 976	9.79 367	9.89 983	0.10 017	9.89 385	33
28	9.78 412	9.88 446	0.11 554	9.89 966	9.79 383	9.90 009	0.09 991	9.89 375	32
29	9.78 428	9.88 472	0.11 528	9.89 956	9.79 399	9.90 035	0.09 965	9.89 364	31
30	9.78 445	9.88 498	0.11 502	9.89 947	9.79 415	9.90 061	0.09 939	9.89 354	**30**
31	9.78 461	9.88 524	0.11 476	9.89 937	9.79 431	9.90 086	0.09 914	9.89 344	29
32	9.78 478	9.88 550	0.11 450	9.89 927	9.79 447	9.90 112	0.09 888	9.89 334	28
33	9.78 494	9.88 577	0.11 423	9.89 918	9.79 463	9.90 138	0.09 862	9.89 324	27
34	9.78 510	9.88 603	0.11 397	9.89 908	9.79 478	9.90 164	0.09 836	9.89 314	26
35	9.78 527	9.88 629	0.11 371	9.89 898	9.79 494	9.90 190	0.09 810	9.89 304	25
36	9.78 543	9.88 655	0.11 345	9.89 888	9.79 510	9.90 216	0.09 784	9.89 294	24
37	9.78 560	9.88 681	0.11 319	9.89 879	9.79 526	9.90 242	0.09 758	9.89 284	23
38	9.78 576	9.88 707	0.11 293	9.89 869	9.79 542	9.90 268	0.09 732	9.89 274	22
39	9.78 592	9.88 733	0.11 267	9.89 859	9.79 558	9.90 294	0.09 706	9.89 264	21
40	9.78 609	9.88 759	0.11 241	9.89 849	9.79 573	9.90 320	0.09 680	9.89 254	**20**
41	9.78 625	9.88 786	0.11 214	9.89 840	9.79 589	9.90 346	0.09 654	9.89 244	19
42	9.78 642	9.88 812	0.11 188	9.89 830	9.79 605	9.90 371	0.09 629	9.89 233	18
43	9.78 658	9.88 838	0.11 162	9.89 820	9.79 621	9.90 397	0.09 603	9.89 223	17
44	9.78 674	9.88 864	0.11 136	9.89 810	9.79 636	9.90 423	0.09 577	9.89 213	16
45	9.78 6[illegible]1	9.88 890	0.11 110	9.89 801	9.79 652	9.90 449	0.09 551	9.89 203	15
46	9.78 707	9.88 916	0.11 084	9.89 791	9.79 668	9.90 475	0.09 525	9.89 193	14
47	9.78 723	9.88 942	0.11 058	9.89 781	9.79 684	9.90 501	0.09 499	9.89 183	13
48	9.78 739	9.88 968	0.11 032	9.89 771	9.79 699	9.90 527	0.09 473	9.89 173	12
49	9.78 756	9.88 994	0.11 006	9.89 761	9.79 715	9.90 553	0.09 447	9.89 162	11
50	9.78 772	9.89 020	0.10 980	9.89 752	9.79 731	9.90 578	0.09 422	9.89 152	**10**
51	9.78 788	9.89 046	0.10 954	9.89 742	9.79 746	9.90 604	0.09 396	9.89 142	9
52	9.78 805	9.89 073	0.10 927	9.89 732	9.79 762	9.90 630	0.09 370	9.89 132	8
53	9.78 821	9.89 099	0.10 901	9.89 722	9.79 778	9.90 656	0.09 344	9.89 122	7
54	9.78 837	9.89 125	0.10 875	9.89 712	9.79 793	9.90 682	0.09 318	9.89 112	6
55	9.78 853	9.89 151	0.10 849	9.89 702	9.79 809	9.90 708	0.09 292	9.89 101	5
56	9.78 869	9.89 177	0.10 823	9.89 693	9.79 825	9.90 734	0.09 266	9.89 091	4
57	9.78 886	9.89 203	0.10 797	9.89 683	9.79 840	9.90 759	0.09 241	9.89 081	3
58	9.78 902	9.89 229	0.10 771	9.89 673	9.79 856	9.90 785	0.09 215	9.89 071	2
59	9.78 918	9.89 255	0.10 745	9.89 663	9.79 872	9.90 811	0.09 189	9.89 060	1
60	9.78 934	9.89 281	0.10 719	9.89 653	9.79 887	9.90 837	0.09 163	9.89 050	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	52°				51°				′

′	39°				40°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.79 887	9.90 837	0.09 163	9.89 050	9.80 807	9.92 381	0.07 619	9.88 425	**60**
1	9.79 903	9.90 863	0.09 137	9.89 040	9.80 822	9.92 407	0.07 593	9.88 415	59
2	9.79 918	9.90 889	0.09 111	9.89 030	9.80 837	9.92 433	0.07 567	9.88 404	58
3	9.79 934	9.90 914	0.09 086	9.89 020	9.80 852	9.92 458	0.07 542	9.88 394	57
4	9.79 950	9.90 940	0.09 060	9.89 009	9.80 867	9.92 484	0.07 516	9.88 383	56
5	9.79 965	9.90 966	0.09 034	9.88 999	9.80 882	9.92 510	0.07 490	9.88 372	55
6	9.79 981	9.90 992	0.09 008	9.88 989	9.80 897	9.92 535	0.07 465	9.88 362	54
7	9.79 996	9.91 018	0.08 982	9.88 978	9.80 912	9.92 561	0.07 439	9.98 351	53
8	9.80 012	9.91 043	0.08 957	9.88 968	9.80 927	9.92 587	0.07 413	9.88 340	52
9	9.80 027	9.91 069	0.08 931	9.88 958	9.80 942	9.92 612	0.07 388	9.88 330	51
10	9.80 043	9.91 095	0.08 905	9.88 948	9.80 957	9.92 638	0.07 362	9.88 319	**50**
11	9.80 058	9.91 121	0.08 879	9.88 937	9.80 972	9.92 663	0.07 337	9.88 308	49
12	9.80 074	9.91 147	0.08 853	9.88 927	9.80 987	9.92 689	0.07 311	9.88 298	48
13	9.80 089	9.91 172	0.08 828	9.88 917	9.81 002	9.92 715	0.07 285	9.88 287	47
14	9.80 105	9.91 198	0.08 802	9.88 906	9.81 017	9.92 740	0.07 260	9.88 276	46
15	9.80 120	9.91 224	0.08 776	9.88 896	9.81 032	9.92 766	0.07 234	9.88 266	45
16	9.80 136	9.91 250	0.08 750	9.88 886	9.81 047	9.92 792	0.07 208	9.88 255	44
17	9.80 151	9.91 276	0.08 724	9.88 875	9.81 061	9.92 817	0.07 183	9.88 244	43
18	9.80 166	9.91 301	0.08 699	9.88 865	9.81 076	9.92 843	0.07 157	9.88 234	42
19	9.80 182	9.91 327	0.08 673	9.88 855	9.81 091	9.92 868	0.07 132	9.88 223	41
20	9.80 197	9.91 353	0.08 647	9.88 844	9.81 106	9.92 894	0.07 106	9.88 212	**40**
21	9.80 213	9.91 379	0.08 621	9.88 834	9.81 121	9.92 920	0.07 080	9.88 201	39
22	9.80 228	9.91 404	0.08 596	9.88 824	9.81 136	9.92 945	0.07 055	9.88 191	38
23	9.80 244	9.91 430	0.08 570	9.88 813	9.81 151	9.92 971	0.07 029	9.88 180	37
24	9.80 259	9.91 456	0.08 544	9.88 803	9.81 166	9.92 996	0.07 004	9.88 169	36
25	9.80 274	9.91 482	0.08 518	9.88 793	9.81 180	9.93 022	0.06 978	9.88 158	35
26	9.80 290	9.91 507	0.08 493	9.88 782	9.81 195	9.93 048	0.06 952	9.88 148	34
27	9.80 305	9.91 533	0.08 467	9.88 772	9.81 210	9.93 073	0.06 927	9.88 137	33
28	9.80 320	9.91 559	0.08 441	9.88 761	9.81 225	9.93 099	0.06 901	9.88 126	32
29	9.80 336	9.91 585	0.08 415	9.88 751	9.81 240	9.93 124	0.06 876	9.88 115	31
30	9.80 351	9.91 610	0.08 390	9.88 741	9.81 254	9.93 150	0.06 850	9.88 105	**30**
31	9.80 366	9.91 636	0.08 364	9.88 730	9.81 269	9.93 175	0.06 825	9.88 094	29
32	9.80 382	9.91 662	0.08 338	9.88 720	9.81 284	9.93 201	0.06 799	9.88 083	28
33	9.80 397	9.91 688	0.08 312	9.88 709	9.81 299	9.93 227	0.06 773	9.88 072	27
34	9.80 412	9.91 713	0.08 287	9.88 699	9.81 314	9.93 252	0.06 748	9.88 061	26
35	9.80 428	9.91 739	0.08 261	9.88 688	9.81 328	9.93 278	0.06 722	9.88 051	25
36	9.80 443	9.91 765	0.08 235	9.88 678	9.81 343	9.93 303	0.06 697	9.88 040	24
37	9.80 458	9.91 791	0.08 209	9.88 668	9.81 358	9.93 329	0.06 671	9.88 029	23
38	9.80 473	9.91 816	0.08 184	9.88 657	9.81 372	9.93 354	0.06 646	9.88 018	22
39	9.80 489	9.91 842	0.08 158	9.88 647	9.81 387	9.93 380	0.06 620	9.88 007	21
40	9.80 504	9.91 868	0.08 132	9.88 636	9.81 402	9.93 406	0.06 594	9.87 996	**20**
41	9.80 519	9.91 893	0.08 107	9.88 626	9.81 417	9.93 431	0.06 569	9.87 985	19
42	9.80 534	9.91 919	0.08 081	9.88 615	9.81 431	9.93 457	0.06 543	9.87 975	18
43	9.80 550	9.91 945	0.08 055	9.88 605	9.81 446	9.93 482	0.06 518	9.87 964	17
44	9.80 565	9.91 971	0.08 029	9.88 594	9.81 461	9.93 508	0.06 492	9.87 953	16
45	9.80 580	9.91 996	0.08 004	9.88 584	9.81 475	9.93 533	0.06 467	9.87 942	15
46	9.80 595	9.92 022	0.07 978	9.88 573	9.81 490	9.93 559	0.06 441	9.87 931	14
47	9.80 610	9.92 048	0.07 952	9.88 563	9.81 505	9.93 584	0.06 416	9.87 920	13
48	9.80 625	9.92 073	0.07 927	9.88 552	9.81 519	9.93 610	0.06 390	9.87 909	12
49	9.80 641	9.92 099	0.07 901	9.88 542	9.81 534	9.93 636	0.06 364	9.87 898	11
50	9.80 656	9.92 125	0.07 875	9.88 531	9.81 549	9.93 661	0.06 339	9.87 887	**10**
51	9.80 671	9.92 150	0.07 850	9.88 521	9.81 563	9.93 687	0.06 313	9.87 877	9
52	9.80 686	9.92 176	0.07 824	9.88 510	9.81 578	9.93 712	0.06 288	9.87 866	8
53	9.80 701	9.92 202	0.07 798	9.88 499	9.81 592	9.93 738	0.06 262	9.87 855	7
54	9.80 716	9.92 227	0.07 773	9.88 489	9.81 607	9.93 763	0.06 237	9.87 844	6
55	9.80 731	9.92 253	0.07 747	9.88 478	9.81 622	9.93 789	0.06 211	9.87 833	5
56	9.80 746	9.92 279	0.07 721	9.88 468	9.81 636	9.93 814	0.06 186	9.87 822	4
57	9.80 762	9.92 304	0.07 696	9.88 457	9.81 651	9.93 840	0.06 160	9.87 811	3
58	9.80 777	9.92 330	0.07 670	9.88 447	9.81 665	9.93 865	0.06 135	9.87 800	2
59	9.80 792	9.92 356	0.07 644	9.88 436	9.81 680	9.93 891	0.06 109	9.87 789	1
60	9.80 807	9.92 381	0.07 619	9.88 425	9.81 694	9.93 916	0.06 084	9.87 778	**0**
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	50°				49°				′

′	41°				42°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.81 694	9.93 916	0.06 084	9.87 778	9.82 551	9.95 444	0.04 556	9.87 107	60
1	9.81 709	9.93 942	0.06 058	9.87 767	9.82 565	9.95 469	0.04 531	9.87 096	59
2	9.81 723	9.93 967	0.06 033	9.87 756	9.82 579	9.95 495	0.04 505	9.87 085	58
3	9.81 738	9.93 993	0.06 007	9.87 745	9.82 593	9.95 520	0.04 480	9.87 073	57
4	9.81 752	9.94 018	0.05 982	9.87 734	9.82 607	9.95 545	0.04 455	9.87 062	56
5	9.81 767	9.94 044	0.05 956	9.87 723	9.82 621	9.95 571	0.04 429	9.87 050	55
6	9.81 781	9.94 069	0.05 931	9.87 712	9.82 635	9.95 596	0.04 404	9.87 039	54
7	9.81 796	9.94 095	0.05 905	9.87 701	9.82 649	9.95 622	0.04 378	9.87 028	53
8	9.81 810	9.94 120	0.05 880	9.87 690	9.82 663	9.95 647	0.04 353	9.87 016	52
9	9.81 825	9.94 146	0.05 854	9.87 679	9.82 677	9.95 672	0.04 328	9.87 005	51
10	9.81 839	9.94 171	0.05 829	9.87 668	9.82 691	9.95 698	0.04 302	9.86 993	50
11	9.81 854	9.94 197	0.05 803	9.87 657	9.82 705	9.95 723	0.04 277	9.86 982	49
12	9.81 868	9.94 222	0.05 778	9.87 646	9.82 719	9.95 748	0.04 252	9.86 970	48
13	9.81 882	9.94 248	0.05 752	9.87 635	9.82 733	9.95 774	0.04 226	9.86 959	47
14	9.81 897	9.94 273	0.05 727	9.87 624	9.82 747	9.95 799	0.04 201	9.86 947	46
15	9.81 911	9.94 299	0.05 701	9.87 613	9.82 761	9.95 825	0.04 175	9.86 936	45
16	9.81 926	9.94 324	0.05 676	9.87 601	9.82 775	9.95 850	0.04 150	9.86 924	44
17	9.81 940	9.94 350	0.05 650	9.87 590	9.82 788	9.95 875	0.04 125	9.86 913	43
18	9.81 955	9.94 375	0.05 625	9.87 579	9.82 802	9.95 901	0.04 099	9.86 902	42
19	9.81 969	9.94 401	0.05 599	9.87 568	9.82 816	9.95 926	0.04 074	9.86 890	41
20	9.81 983	9.94 426	0.05 574	9.87 557	9.82 830	9.95 952	0.04 048	9.86 879	40
21	9.81 998	9.94 452	0.05 548	9.87 546	9.82 844	9.95 977	0.04 023	9.86 867	39
22	9.82 012	9.94 477	0.05 523	9.87 535	9.82 858	9.96 002	0.03 998	9.86 855	38
23	9.82 026	9.94 503	0.05 497	9.87 524	9.82 872	9.96 028	0.03 972	9.86 844	37
24	9.82 041	9.94 528	0.05 472	9.87 513	9.82 885	9.96 053	0.03 947	9.86 832	36
25	9.82 055	9.94 554	0.05 446	9.87 501	9.82 899	9.96 078	0.03 922	9.86 821	35
26	9.82 069	9.94 579	0.05 421	9.87 490	9.82 913	9.96 104	0.03 896	9.86 809	34
27	9.82 084	9.94 604	0.05 396	9.87 479	9.82 927	9.96 129	0.03 871	9.86 798	33
28	9.82 098	9.94 630	0.05 370	9.87 468	9.82 941	9.96 155	0.03 845	9.86 786	32
29	9.82 112	9.94 655	0.05 345	9.87 457	9.82 955	9.96 180	0.03 820	9.86 775	31
30	9.82 126	9.94 681	0.05 319	9.87 446	9.82 968	9.96 205	0.03 795	9.86 763	30
31	9.82 141	9.94 706	0.05 294	9.87 434	9.82 982	9.96 231	0.03 769	9.86 752	29
32	9.82 155	9.94 732	0.05 268	9.87 423	9.82 996	9.96 256	0.03 744	9.86 740	28
33	9.82 169	9.94 757	0.05 243	9.87 412	9.83 010	9.96 281	0.03 719	9.86 728	27
34	9.82 184	9.94 783	0.05 217	9.87 401	9.83 023	9.96 307	0.03 693	9.86 717	26
35	9.82 198	9.94 808	0.05 192	9.87 390	9.83 037	9.96 332	0.03 668	9.86 705	25
36	9.82 212	9.94 834	0.05 166	9.87 378	9.83 051	9.96 357	0.03 643	9.86 694	24
37	9.82 226	9.94 859	0.05 141	9.87 367	9.83 065	9.96 383	0.03 617	9.86 682	23
38	9.82 240	9.94 884	0.05 116	9.87 356	9.83 078	9.96 408	0.03 592	9.86 670	22
39	9.82 255	9.94 910	0.05 090	9.87 345	9.83 092	9.96 433	0.03 567	9.86 659	21
40	9.82 269	9.94 935	0.05 065	9.87 334	9.83 106	9.96 459	0.03 541	9.86 647	20
41	9.82 283	9.94 961	0.05 039	9.87 322	9.83 120	9.96 484	0.03 516	9.86 635	19
42	9.82 297	9.94 986	0.05 014	9.87 311	9.83 133	9.96 510	0.03 490	9.86 624	18
43	9.82 311	9.95 012	0.04 988	9.87 300	9.83 147	9.96 535	0.03 465	9.86 612	17
44	9.82 326	9.95 037	0.04 963	9.87 288	9.83 161	9.96 560	0.03 440	9.86 600	16
45	9.82 340	9.95 062	0.04 938	9.87 277	9.83 174	9.96 586	0.03 414	9.86 589	15
46	9.82 354	9.95 088	0.04 912	9.87 266	9.83 188	9.96 611	0.03 389	9.86 577	14
47	9.82 368	9.95 113	0.04 887	9.87 255	9.83 202	9.96 636	0.03 364	9.86 565	13
48	9.82 382	9.95 139	0.04 861	9.87 243	9.83 215	9.96 662	0.03 338	9.86 554	12
49	9.82 396	9.95 164	0.04 836	9.87 232	9.83 229	9.96 687	0.03 313	9.86 542	11
50	9.82 410	9.95 190	0.04 810	9.87 221	9.83 242	9.96 712	0.03 288	9.86 530	10
51	9.82 424	9.95 215	0.04 785	9.87 209	9.83 256	9.96 738	0.03 262	9.86 518	9
52	9.82 439	9.95 240	0.04 760	9.87 198	9.83 270	9.96 763	0.03 237	9.86 507	8
53	9.82 453	9.95 266	0.04 734	9.87 187	9.83 283	9.96 788	0.03 212	9.86 495	7
54	9.82 467	9.95 291	0.04 709	9.87 175	9.83 297	9.96 814	0.03 186	9.86 483	6
55	9.82 481	9.95 317	0.04 683	9.87 164	9.83 310	9.96 839	0.03 161	9.86 472	5
56	9.82 495	9.95 342	0 04 658	9.87 153	9.83 324	9.96 864	0.03 136	9.86 460	4
57	9.82 509	9.95 368	0.04 632	9.87 141	9.83 338	9.96 890	0.03 110	9.86 448	3
58	9.82 523	9.95 393	0.04 607	9 87 130	9.83 351	9.96 915	0.03 085	9.86 436	2
59	9.82 537	9.95 418	0.04 582	9.87 119	9.83 365	9.96 940	0.03 060	9.86 425	1
60	9.82 551	9.95 444	0.04 556	9.87 107	9.83 378	9.96 966	0.03 034	9.86 413	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	48°				47°				′

′	43°				44°				′
	log sin	log tan	log cot	log cos	log sin	log tan	log cot	log cos	
0	9.83 378	9.96 966	0.03 034	9.86 413	9.84 177	9.98 484	0.01 516	9.85 693	60
1	9.83 392	9.96 991	0.03 009	9.86 401	9.84 190	9.98 509	0.01 491	9.85 681	59
2	9.83 405	9.97 016	0.02 984	9.86 389	9.84 203	9.98 534	0.01 466	9.85 669	58
3	9.83 419	9.97 042	0.02 958	9.86 377	9.84 216	9.98 560	0.01 440	9.85 657	57
4	9.83 432	9.97 067	0.02 933	9.86 366	9.84 229	9.98 585	0.01 415	9.85 645	56
5	9.83 446	9.97 092	0.02 908	9.86 354	9.84 242	9.98 610	0.01 390	9.85 632	55
6	9.83 459	9.97 118	0.02 882	9.86 342	9.84 255	9.98 635	0.01 365	9.85 620	54
7	9.83 473	9.97 143	0.02 857	9.86 330	9.84 269	9.98 661	0.01 339	9.85 608	53
8	9.83 486	9.97 168	0.02 832	9.86 318	9.84 282	9.98 686	0.01 314	9.85 596	52
9	9.83 500	9.97 193	0.02 807	9.86 306	9.84 295	9.98 711	0.01 289	9.85 583	51
10	9.83 513	9.97 219	0.02 781	9.86 295	9.84 308	9.98 737	0.01 263	9.85 571	50
11	9.83 527	9.97 244	0.02 756	9.86 283	9.84 321	9.98 762	0.01 238	9.85 559	49
12	9.83 540	9.97 269	0.02 731	9.86 271	9.84 334	9.98 787	0.01 213	9.85 547	48
13	9.83 554	9.97 295	0.02 705	9.86 259	9.84 347	9.98 812	0.01 188	9.85 534	47
14	9.83 567	9.97 320	0.02 680	9.86 247	9.84 360	9.98 838	0.01 162	9.85 522	46
15	9.83 581	9.97 345	0.02 655	9.86 235	9.84 373	9.98 863	0.01 137	9.85 510	45
16	9.83 594	9.97 371	0.02 629	9.86 223	9.84 385	9.98 888	0.01 112	9.85 497	44
17	9.83 608	9.97 396	0.02 604	9.86 211	9.84 398	9.98 913	0.01 087	9.85 485	43
18	9.83 621	9.97 421	0.02 579	9.86 200	9.84 411	9.98 939	0.01 061	9.85 473	42
19	9.83 634	9.97 447	0.02 553	9.86 188	9.84 424	9.98 964	0.01 036	9.85 460	41
20	9.83 648	9.97 472	0.02 528	9.86 176	9.84 437	9.98 989	0.01 011	9.85 448	40
21	9.83 661	9.97 497	0.02 503	9.86 164	9.84 450	9.99 015	0.00 985	9.85 436	39
22	9.83 674	9.97 523	0.02 477	9.86 152	9.84 463	9.99 040	0.00 960	9.85 423	38
23	9.83 688	9.97 548	0.02 452	9.86 140	9.84 476	9.99 065	0.00 935	9.85 411	37
24	9.83 701	9.97 573	0.02 427	9.86 128	9.84 489	9.99 090	0.00 910	9.85 399	36
25	9.83 715	9.97 598	0.02 402	9.86 116	9.84 502	9.99 116	0.00 884	9.85 386	35
26	9.83 728	9.97 624	0.02 376	9.86 104	9.84 515	9.99 141	0.00 859	9.85 374	34
27	9.83 741	9.97 649	0.02 351	9.86 092	9.84 528	9.99 166	0.00 834	9.85 361	33
28	9.83 755	9.97 674	0.02 326	9.86 080	9.84 540	9.99 191	0.00 809	9.85 349	32
29	9.83 768	9.97 700	0.02 300	9.86 068	9.84 553	9.99 217	0.00 783	9.85 337	31
30	9.83 781	9.97 725	0.02 275	9.86 056	9.84 566	9.99 242	0.00 758	9.85 324	30
31	9.83 795	9.97 750	0.02 250	9.86 044	9.84 579	9.99 267	0.00 733	9.85 312	29
32	9.83 808	9.97 776	0.02 224	9.86 032	9.84 592	9.99 293	0.00 707	9.85 299	28
33	9.83 821	9.97 801	0.02 199	9.86 020	9.84 605	9.99 318	0.00 682	9.85 287	27
34	9.83 834	9.97 826	0.02 174	9.86 008	9.84 618	9.99 343	0.00 657	9.85 274	26
35	9.83 848	9.97 851	0.02 149	9.85 996	9.84 630	9.99 368	0.00 632	9.85 262	25
36	9.83 861	9.97 877	0.02 123	9.85 984	9.84 643	9.99 394	0.00 606	9.85 250	24
37	9.83 874	9.97 902	0.02 098	9.85 972	9.84 656	9.99 419	0.00 581	9.85 237	23
38	9.83 887	9.97 927	0.02 073	9.85 960	9.84 669	9.99 444	0.00 556	9.85 225	22
39	9.83 901	9.97 953	0.02 047	9.85 948	9.84 682	9.99 469	0.00 531	9.85 212	21
40	9.83 914	9.97 978	0.02 022	9.85 936	9.84 694	9.99 495	0.00 505	9.85 200	20
41	9.83 927	9.98 003	0.01 997	9.85 924	9.84 707	9.99 520	0.00 480	9.85 187	19
42	9.83 940	9.98 029	0.01 971	9.85 912	9.84 720	9.99 545	0.00 455	9.85 175	18
43	9.83 954	9.98 054	0.01 946	9.85 900	9.84 733	9.99 570	0.00 430	9.85 162	17
44	9.83 967	9.98 079	0.01 921	9.85 888	9.84 745	9.99 596	0.00 404	9.85 150	16
45	9.83 980	9.98 104	0.01 896	9.85 876	9.84 758	9.99 621	0.00 379	9.85 137	15
46	9.83 993	9.98 130	0.01 870	9.85 864	9.84 771	9.99 646	0.00 354	9.85 125	14
47	9.84 006	9.98 155	0.01 845	9.85 851	9.84 784	9.99 672	0.00 328	9.85 112	13
48	9.84 020	9.98 180	0.01 820	9.85 839	9.84 796	9.99 697	0.00 303	9.85 100	12
49	9.84 033	9.98 206	0.01 794	9.85 827	9.84 809	9.99 722	0.00 278	9.85 087	11
50	9.84 046	9.98 231	0.01 769	9.85 815	9.84 822	9.99 747	0.00 253	9.85 074	10
51	9.84 059	9.98 256	0.01 744	9.85 803	9.84 835	9.99 773	0.00 227	9.85 062	9
52	9.84 072	9.98 281	0.01 719	9.85 791	9.84 847	9.99 798	0.00 202	9.85 049	8
53	9.84 085	9.98 307	0.01 693	9.85 779	9.84 860	9.99 823	0.00 177	9.85 037	7
54	9.84 098	9.98 332	0.01 668	9.85 766	9.84 873	9.99 848	0.00 152	9.85 024	6
55	9.84 112	9.98 357	0.01 643	9.85 754	9.84 885	9.99 874	0.00 126	9.85 012	5
56	9.84 125	9.98 383	0.01 617	9.85 742	9.84 898	9.99 899	0.00 101	9.84 999	4
57	9.84 138	9.98 408	0.01 592	9.85 730	9.84 911	9.99 924	0.00 076	9.84 986	3
58	9.84 151	9.98 433	0.01 567	9.85 718	9.84 923	9.99 949	0.00 051	9.84 974	2
59	9.84 164	9.98 458	0.01 542	9.85 706	9.84 936	9.99 975	0.00 025	9.84 961	1
60	9.84 177	9.98 484	0.01 516	9.85 693	9.84 949	0.00 000	0.00 000	9.84 949	0
	log cos	log cot	log tan	log sin	log cos	log cot	log tan	log sin	
′	46°				45°				′

TABLE III

NATURAL SINES

AND

COSINES

′	**0°**		**1°**		**2°**		**3°**		**4°**		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	0000	1.000	0175	9998	0349	9994	0523	9986	0698	9976	**60**
1	0003	1.000	0177	9998	0352	9994	0526	9986	0700	9975	59
2	0006	1.000	0180	9998	0355	9994	0529	9986	0703	9975	58
3	0009	1.000	0183	9998	0358	9994	0532	9986	0706	9975	57
4	0012	1.000	0186	9998	0361	9993	0535	9986	0709	9975	56
5	0015	1.000	0189	9998	0364	9993	0538	9986	0712	9975	55
6	0017	1.000	0192	9998	0366	9993	0541	9985	0715	9974	54
7	0020	1.000	0195	9998	0369	9993	0544	9985	0718	9974	53
8	0023	1.000	0198	9998	0372	9993	0547	9985	0721	9974	52
9	0026	1.000	0201	9998	0375	9993	0550	9985	0724	9974	51
10	0029	1.000	0204	9998	0378	9993	0552	9985	0727	9974	**50**
11	0032	1.000	0207	9998	0381	9993	0555	9985	0729	9973	49
12	0035	1.000	0209	9998	0384	9993	0558	9984	0732	9973	48
13	0038	1.000	0212	9998	0387	9993	0561	9984	0735	9973	47
14	0041	1.000	0215	9998	0390	9992	0564	9984	0738	9973	46
15	0044	1.000	0218	9998	0393	9992	0567	9984	0741	9973	45
16	0047	1.000	0221	9998	0396	9992	0570	9984	0744	9972	44
17	0049	1.000	0224	9997	0398	9992	0573	9984	0747	9972	43
18	0052	1.000	0227	9997	0401	9992	0576	9983	0750	9972	42
19	0055	1.000	0230	9997	0404	9992	0579	9983	0753	9972	41
20	0058	1.000	0233	9997	0407	9992	0581	9983	0756	9971	**40**
21	0061	1.000	0236	9997	0410	9992	0584	9983	0758	9971	39
22	0064	1.000	0239	9997	0413	9991	0587	9983	0761	9971	38
23	0067	1.000	0241	9997	0416	9991	0590	9983	0764	9971	37
24	0070	1.000	0244	9997	0419	9991	0593	9982	0767	9971	36
25	0073	1.000	0247	9997	0422	9991	0596	9982	0770	9970	35
26	0076	1.000	0250	9997	0425	9991	0599	9982	0773	9970	34
27	0079	1.000	0253	9997	0427	9991	0602	9982	0776	9970	33
28	0081	1.000	0256	9997	0430	9991	0605	9982	0779	9970	32
29	0084	1.000	0259	9997	0433	9991	0608	9982	0782	9969	31
30	0087	1.000	0262	9997	0436	9990	0610	9981	0785	9969	**30**
31	0090	1.000	0265	9996	0439	9990	0613	9981	0787	9969	26
32	0093	1.000	0268	9996	0442	9990	0616	9981	0790	9969	28
33	0096	1.000	0270	9996	0445	9990	0619	9981	0793	9968	27
34	0099	1.000	0273	9996	0448	9990	0622	9981	0796	9968	26
35	0102	9999	0276	9996	0451	9990	0625	9980	0799	9968	25
36	0105	9999	0279	9996	0454	9990	0628	9980	0802	9968	
37	0108	9999	0282	9996	0457	9990	0631	9980	0805	9968	24
38	0111	9999	0285	9996	0459	9989	0634	9980	0808	9967	23
39	0113	9999	0288	9996	0462	9989	0637	9980	0811	9967	22 21
40	0116	9999	0291	9996	0465	9989	0640	9980	0814	9967	**20**
41	0119	9999	0294	9996	0468	9989	0642	9979	0816	9967	19
42	0122	9999	0297	9996	0471	9989	0645	9979	0819	9966	18
43	0125	9999	0300	9996	0474	9989	0648	9979	0822	9966	17
44	0128	9999	0302	9995	0477	9989	0651	9979	0825	9966	16
45	0131	9999	0305	9995	0480	9988	0654	9979	0828	9966	15
46	0134	9999	0308	9995	0483	9988	0657	9978	0831	9965	14
47	0137	9999	0311	9995	0486	9988	0660	9978	0834	9965	13
48	0140	9999	0314	9995	0488	9988	0663	9978	0837	9965	12
49	0143	9999	0317	9995	0491	9988	0666	9978	0840	9965	11
50	0145	9999	0320	9995	0494	9988	0669	9978	0843	9964	**10**
51	0148	9999	0323	9995	0497	9988	0671	9977	0845	9964	9
52	0151	9999	0326	9995	0500	9987	0674	9977	0848	9964	8
53	0154	9999	0329	9995	0503	9987	0677	9977	0851	9964	7
54	0157	9999	0332	9995	0506	9987	0680	9977	0854	9963	6
55	0160	9999	0334	9994	0509	9987	0683	9977	0857	9963	5
56	0163	9999	0337	9994	0512	9987	0686	9976	0860	9963	4
57	0166	9999	0340	9994	0515	9987	0689	9976	0863	9963	3
58	0169	9999	0343	9994	0518	9987	0692	9976	0866	9962	2
59	0172	9999	0346	9994	0520	9986	0695	9976	0869	9962	1
60	0175	9999	0349	9994	0523	9986	0698	9976	0872	9962	**0**
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	**89°**		**88°**		**87°**		**86°**		**85°**		′

′	5°		6°		7°		8°		9°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	0872	9962	1045	9945	1219	9925	1392	9903	1564	9877	**60**
1	0874	9962	1048	9945	1222	9925	1395	9902	1567	9876	59
2	0877	9961	1051	9945	1224	9925	1397	9902	1570	9876	58
3	0880	9961	1054	9944	1227	9924	1400	9901	1573	9876	57
4	0883	9961	1057	9944	1230	9924	1403	9901	1576	9875	56
5	0886	9961	1060	9944	1233	9924	1406	9901	1579	9875	55
6	0889	9960	1063	9943	1236	9923	1409	9900	1582	9874	54
7	0892	9960	1066	9943	1239	9923	1412	9900	1584	9874	53
8	0895	9960	1068	9943	1241	9923	1415	9899	1587	9873	52
9	0898	9960	1071	9942	1245	9922	1418	9899	1590	9873	51
10	0901	9959	1074	9942	1248	9922	1421	9899	1593	9872	**50**
11	0903	9959	1077	9942	1250	9922	1423	9898	1596	9872	49
12	0906	9959	1080	9942	1253	9921	1426	9898	1599	9871	48
13	0909	9959	1083	9941	1256	9921	1429	9897	1602	9871	47
14	0912	9958	1086	9941	1259	9920	1432	9897	1605	9870	46
15	0915	9958	1089	9941	1262	9920	1435	9897	1607	9870	45
16	0918	9958	1092	9940	1265	9920	1438	9896	1610	9869	44
17	0921	9958	1094	9940	1268	9919	1441	9896	1613	9869	43
18	0924	9957	1097	9940	1271	9919	1444	9895	1616	9869	42
19	0927	9957	1100	9939	1274	9919	1446	9895	1619	9868	41
20	0929	9957	1103	9939	1276	9918	1449	9894	1622	9868	**40**
21	0932	9956	1106	9939	1279	9918	1452	9894	1625	9867	39
22	0935	9956	1109	9938	1282	9917	1455	9894	1628	9867	38
23	0938	9956	1112	9938	1285	9917	1458	9893	1630	9866	37
24	0941	9956	1115	9938	1288	9917	1461	9893	1633	9866	36
25	0944	9955	1118	9937	1291	9916	1464	9892	1636	9865	35
26	0947	9955	1120	9937	1294	9916	1467	9892	1639	9865	34
27	0950	9955	1123	9937	1297	9916	1469	9891	1642	9864	33
28	0953	9955	1126	9936	1299	9915	1472	9891	1645	9864	32
29	0956	9954	1129	9936	1302	9915	1475	9891	1648	9863	31
30	0958	9954	1132	9936	1305	9914	1478	9890	1650	9863	**30**
31	0961	9954	1135	9935	1308	9914	1481	9890	1653	9862	29
32	0964	9953	1138	9935	1311	9914	1484	9889	1656	9862	28
33	0967	9953	1141	9935	1314	9913	1487	9889	1659	9861	27
34	0970	9953	1144	9934	1317	9913	1490	9888	1662	9861	26
35	0973	9953	1146	9934	1320	9913	1492	9888	1665	9860	25
36	0976	9952	1149	9934	1323	9912	1495	9888	1668	9860	24
37	0979	9952	1152	9933	1325	9912	1498	9887	1671	9859	23
38	0982	9952	1155	9933	1328	9911	1501	9887	1673	9859	22
39	0985	9951	1158	9933	1331	9911	1504	9886	1676	9859	21
40	0987	9951	1161	9932	1334	9911	1507	9886	1679	9858	**20**
41	0990	9951	1164	9932	1337	9910	1510	9885	1682	9858	19
42	0993	9951	1167	9932	1340	9910	1513	9885	1685	9857	18
43	0996	9950	1170	9931	1343	9909	1515	9884	1688	9857	17
44	0999	9950	1172	9931	1346	9909	1518	9884	1691	9856	16
45	1002	9950	1175	9931	1349	9909	1521	9884	1693	9856	15
46	1005	9949	1178	9930	1351	9908	1524	9883	1696	9855	14
47	1008	9949	1181	9930	1354	9908	1527	9883	1699	9855	13
48	1011	9949	1184	9930	1357	9907	1530	9882	1702	9854	12
49	1013	9949	1187	9929	1360	9907	1533	9882	1705	9854	11
50	1016	9948	1190	9929	1363	9907	1536	9881	1708	9853	**10**
51	1019	9948	1193	9929	1366	9906	1538	9881	1711	9853	9
52	1022	9948	1196	9928	1369	9906	1541	9880	1714	9852	8
53	1025	9947	1198	9928	1372	9905	1544	9880	1716	9852	7
54	1028	9947	1201	9928	1374	9905	1547	9880	1719	9851	6
55	1031	9947	1204	9927	1377	9905	1550	9879	1722	9851	5
56	1034	9946	1207	9927	1380	9904	1553	9879	1725	9850	4
57	1037	9946	1210	9927	1383	9904	1556	9878	1728	9850	3
58	1039	9946	1213	9926	1386	9903	1559	9878	1731	9849	2
59	1042	9946	1216	9926	1389	9903	1561	9877	1734	9849	1
60	1045	9945	1219	9925	1392	9903	1564	9877	1736	9848	**0**
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	**84°**		**83°**		**82°**		**81°**		**80°**		′

′	10°		11°		12°		13°		14°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	1736	9848	1908	9816	2079	9781	2250	9744	2419	9703	60
1	1739	9848	1911	9816	2082	9781	2252	9743	2422	9702	59
2	1742	9847	1914	9815	2085	9780	2255	9742	2425	9702	58
3	1745	9847	1917	9815	2088	9780	2258	9742	2428	9701	57
4	1748	9846	1920	9814	2090	9779	2261	9741	2431	9700	56
5	1751	9846	1922	9813	2093	9778	2264	9740	2433	9699	55
6	1754	9845	1925	9813	2096	9778	2267	9740	2436	9699	54
7	1757	9845	1928	9812	2099	9777	2269	9739	2439	9698	53
8	1759	9844	1931	9812	2102	9777	2272	9738	2442	9697	52
9	1762	9843	1934	9811	2105	9776	2275	9738	2445	9697	51
10	1765	9843	1937	9811	2108	9775	2278	9737	2447	9696	50
11	1768	9842	1939	9810	2110	9775	2281	9736	2450	9695	49
12	1771	9842	1942	9810	2113	9774	2284	9736	2453	9694	48
13	1774	9841	1945	9809	2116	9774	2286	9735	2456	9694	47
14	1777	9841	1948	9808	2119	9773	2289	9734	2459	9693	46
15	1779	9840	1951	9808	2122	9772	2292	9734	2462	9692	45
16	1782	9840	1954	9807	2125	9772	2295	9733	2464	9692	44
17	1785	9839	1957	9807	2127	9771	2298	9732	2467	9691	43
18	1788	9839	1959	9806	2130	9770	2300	9732	2470	9690	42
19	1791	9838	1962	9806	2133	9770	2303	9731	2473	9689	41
20	1794	9838	1965	9805	2136	9769	2306	9730	2476	9689	40
21	1797	9837	1968	9804	2139	9769	2309	9730	2478	9688	39
22	1799	9837	1971	9804	2142	9768	2312	9729	2481	9687	38
23	1802	9836	1974	9803	2145	9767	2315	9728	2484	9687	37
24	1805	9836	1977	9803	2147	9767	2317	9728	2487	9686	36
25	1808	9835	1979	9802	2150	9766	2320	9727	2490	9685	35
26	1811	9835	1982	9802	2153	9765	2323	9726	2493	9684	34
27	1814	9834	1985	9801	2156	9765	2326	9726	2495	9684	33
28	1817	9834	1988	9800	2159	9764	2329	9725	2498	9683	32
29	1819	9833	1991	9800	2162	9764	2332	9724	2501	9682	31
30	1822	9833	1994	9799	2164	9763	2334	9724	2504	9681	30
31	1825	9832	1997	9799	2167	9762	2337	9723	2507	9681	29
32	1828	9831	1999	9798	2170	9762	2340	9722	2509	9680	28
33	1831	9831	2002	9798	2173	9761	2343	9722	2512	9679	27
34	1834	9830	2005	9797	2176	9760	2346	9721	2515	9679	26
35	1837	9830	2008	9796	2179	9760	2349	9720	2518	9678	25
36	1840	9829	2011	9796	2181	9759	2351	9720	2521	9677	24
37	1842	9829	2014	9795	2184	9759	2354	9719	2524	9676	23
38	1845	9828	2016	9795	2187	9758	2357	9718	2526	9676	22
39	1848	9828	2019	9794	2190	9757	2360	9718	2529	9675	21
40	1851	9827	2022	9793	2193	9757	2363	9717	2532	9674	20
41	1854	9827	2025	9793	2196	9756	2366	9716	2535	9673	19
42	1857	9826	2028	9792	2198	9755	2368	9715	2538	9673	18
43	1860	9826	2031	9792	2201	9755	2371	9715	2540	9672	17
44	1862	9825	2034	9791	2204	9754	2374	9714	2543	9671	16
45	1865	9825	2036	9790	2207	9753	2377	9713	2546	9670	15
46	1868	9824	2039	9790	2210	9753	2380	9713	2549	9670	14
47	1871	9823	2042	9789	2213	9752	2383	9712	2552	9669	13
48	1874	9823	2045	9789	2215	9751	2385	9711	2554	9668	12
49	1877	9822	2048	9788	2218	9751	2388	9711	2557	9667	11
50	1880	9822	2051	9787	2221	9750	2391	9710	2560	9667	10
51	1882	9821	2054	9787	2224	9750	2394	9709	2563	9666	9
52	1885	9821	2056	9786	2227	9749	2397	9709	2566	9665	8
53	1888	9820	2059	9786	2230	9748	2399	9708	2569	9665	7
54	1891	9820	2062	9785	2233	9748	2402	9707	2571	9664	6
55	1894	9819	2065	9784	2235	9747	2405	9706	2574	9663	5
56	1897	9818	2068	9784	2238	9746	2408	9706	2577	9662	4
57	1900	9818	2071	9783	2241	9746	2411	9705	2580	9662	3
58	1902	9817	2073	9783	2244	9745	2414	9704	2583	9661	2
59	1905	9817	2076	9782	2247	9744	2416	9704	2585	9660	1
60	1908	9816	2079	9781	2250	9744	2419	9703	2588	9659	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	79°		78°		77°		76°		75°		′

′	15°		16°		17°		18°		19°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	2588	9659	2756	9613	2924	9563	3090	9511	3256	9455	60
1	2591	9659	2759	9612	2926	9562	3093	9510	3258	9454	59
2	2594	9658	2762	9611	2929	9561	3096	9509	3261	9453	58
3	2597	9657	2765	9610	2932	9560	3098	9508	3264	9452	57
4	2599	9656	2768	9609	2935	9560	3101	9507	3267	9451	56
5	2602	9655	2770	9609	2938	9559	3104	9506	3269	9450	55
6	2605	9655	2773	9608	2940	9558	3107	9505	3272	9449	54
7	2608	9654	2776	9607	2943	9557	3110	9504	3275	9449	53
8	2611	9653	2779	9606	2946	9556	3112	9503	3278	9448	52
9	2613	9652	2782	9605	2949	9555	3115	9502	3280	9447	51
10	2616	9652	2784	9605	2952	9555	3118	9502	3283	9446	50
11	2619	9651	2787	9604	2954	9554	3121	9501	3286	9445	49
12	2622	9650	2790	9603	2957	9553	3123	9500	3289	9444	48
13	2625	9649	2793	9602	2960	9552	3126	9499	3291	9443	47
14	2628	9649	2795	9601	2963	9551	3129	9498	3294	9442	46
15	2630	9648	2798	9600	2965	9550	3132	9497	3297	9441	45
16	2633	9647	2801	9600	2968	9549	3134	9496	3300	9440	44
17	2636	9646	2804	9599	2971	9548	3137	9495	3302	9439	43
18	2639	9646	2807	9598	2974	9548	3140	9494	3305	9438	42
19	2642	9645	2809	9597	2977	9547	3143	9493	3308	9437	41
20	2644	9644	2812	9596	2979	9546	3145	9492	3311	9436	40
21	2647	9643	2815	9596	2982	9545	3148	9492	3313	9435	39
22	2650	9642	2818	9595	2985	9544	3151	9491	3316	9434	38
23	2653	9642	2821	9594	2988	9543	3154	9490	3319	9433	37
24	2656	9641	2823	9593	2990	9542	3156	9489	3322	9432	36
25	2658	9640	2826	9592	2993	9542	3159	9488	3324	9431	35
26	2661	9639	2829	9591	2996	9541	3162	9487	3327	9430	34
27	2664	9639	2832	9591	2999	9540	3165	9486	3330	9429	33
28	2667	9638	2835	9590	3002	9539	3168	9485	3333	9428	32
29	2670	9637	2837	9589	3004	9538	3170	9484	3335	9427	31
30	2672	9636	2840	9588	3007	9537	3173	9483	3338	9426	30
31	2675	9636	2843	9587	3010	9536	3176	9482	3341	9425	29
32	2678	9635	2846	9587	3013	9535	3179	9481	3344	9424	28
33	2681	9634	2849	9586	3015	9535	3181	9480	3346	9423	27
34	2684	9633	2851	9585	3018	9534	3184	9480	3349	9423	26
35	2686	9632	2854	9584	3021	9533	3187	9479	3352	9422	25
36	2689	9632	2857	9583	3024	9532	3190	9478	3355	9421	24
37	2692	9631	2860	9582	3026	9531	3192	9477	3357	9420	23
38	2695	9630	2862	9582	3029	9530	3195	9476	3360	9419	22
39	2698	9629	2865	9581	3032	9529	3198	9475	3363	9418	21
40	2700	9628	2868	9580	3035	9528	3201	9474	3365	9417	20
41	2703	9628	2871	9579	3038	9527	3203	9473	3368	9416	19
42	2706	9627	2874	9578	3040	9527	3206	9472	3371	9415	18
43	2709	9626	2876	9577	3043	9526	3209	9471	3374	9414	17
44	2712	9625	2879	9577	3046	9525	3212	9470	3376	9413	16
45	2714	9625	2882	9576	3049	9524	3214	9469	3379	9412	15
46	2717	9624	2885	9575	3051	9523	3217	9468	3382	9411	14
47	2720	9623	2888	9574	3054	9522	3220	9467	3385	9410	13
48	2723	9622	2890	9573	3057	9521	3223	9466	3387	9409	12
49	2726	9621	2893	9572	3060	9520	3225	9466	3390	9408	11
50	2728	9621	2896	9572	3062	9520	3228	9465	3393	9407	10
51	2731	9620	2899	9571	3065	9519	3231	9464	3396	9406	9
52	2734	9619	2901	9570	3068	9518	3234	9463	3398	9405	8
53	2737	9618	2904	9569	3071	9517	3236	9462	3401	9404	7
54	2740	9617	2907	9568	3074	9516	3239	9461	3404	9403	6
55	2742	9617	2910	9567	3076	9515	3242	9460	3407	9402	5
56	2745	9616	2913	9566	3079	9514	3245	9459	3409	9401	4
57	2748	9615	2915	9566	3082	9513	3247	9458	3412	9400	3
58	2751	9614	2918	9565	3085	9512	3250	9457	3415	9399	2
59	2754	9613	2921	9564	3087	9511	3253	9456	3417	9398	1
60	2756	9613	2924	9563	3090	9511	3256	9455	3420	9397	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	74°		73°		72°		71°		70°		′

′	20°		21°		22°		23°		24°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	3420	9397	3584	9336	3746	9272	3907	9205	4067	9135	60
1	3423	9396	3586	9335	3749	9271	3910	9204	4070	9134	59
2	3426	9395	3589	9334	3751	9270	3913	9203	4073	9133	58
3	3428	9394	3592	9333	3754	9269	3915	9202	4075	9132	57
4	3431	9393	3595	9332	3757	9267	3918	9200	4078	9131	56
5	3434	9392	3597	9331	3760	9266	3921	9199	4081	9130	55
6	3437	9391	3600	9330	3762	9265	3923	9198	4083	9128	54
7	3439	9390	3603	9328	3765	9264	3926	9197	4086	9127	53
8	3442	9389	3605	9327	3768	9263	3929	9196	4089	9126	52
9	3445	9388	3608	9326	3770	9262	3931	9195	4091	9125	51
10	3448	9387	3611	9325	3773	9261	3934	9194	4094	9124	50
11	3450	9386	3614	9324	3776	9260	3937	9192	4097	9122	49
12	3453	9385	3616	9323	3778	9259	3939	9191	4099	9121	48
13	3456	9384	3619	9322	3781	9258	3942	9190	4102	9120	47
14	3458	9383	3622	9321	3784	9257	3945	9189	4105	9119	46
15	3461	9382	3624	9320	3786	9255	3947	9188	4107	9118	45
16	3464	9381	3627	9319	3789	9254	3950	9187	4110	9116	44
17	3467	9380	3630	9318	3792	9353	3953	9186	4112	9115	43
18	3469	9379	3633	9317	3795	9252	3955	9184	4115	9114	42
19	3472	9378	3635	9316	3797	9251	3958	9183	4118	9113	41
20	3475	9377	3638	9315	3800	9250	3961	9182	4120	9112	40
21	3478	9376	3641	9314	3803	9249	3963	9181	4123	9110	39
22	3480	9375	3643	9313	3805	9248	3966	9180	4126	9109	38
23	3483	9374	3646	9312	3808	9247	3969	9179	4128	9108	37
24	3486	9373	3649	9311	3811	9245	3971	9178	4131	9107	36
25	3488	9372	3651	9309	3813	9244	3974	9176	4134	9106	35
26	3491	9371	3654	9308	3816	9243	3977	9175	4136	9104	34
27	3494	9370	3657	9307	3819	9242	3979	9174	4139	9103	33
28	3497	9369	3660	9306	3821	9241	3982	9173	4142	9102	32
29	3499	9368	3662	9305	3824	9240	3985	9172	4144	9101	31
30	3502	9367	3665	9304	3827	9239	3987	9171	4147	9100	30
31	3505	9366	3668	9303	3830	9238	3990	9169	4150	9098	29
32	3508	9365	3670	9302	3832	9237	3993	9168	4152	9097	28
33	3510	9364	3673	9301	3835	9235	3995	9167	4155	9096	27
34	3513	9363	3676	9300	3838	9234	3998	9166	4158	9095	26
35	3516	9362	3679	9299	3840	9233	4001	9165	4160	9094	25
36	3518	9361	3681	9298	3843	9232	4003	9164	4163	9092	24
37	3521	9360	3684	9297	3846	9231	4006	9162	4165	9091	23
38	3524	9359	3687	9296	3848	9230	4009	9161	4168	9090	22
39	3527	9358	3689	9295	3851	9229	4011	9160	4171	9089	21
40	3529	9356	3692	9293	3854	9228	4014	9159	4173	9088	20
41	3532	9355	3695	9292	3856	9227	4017	9158	4176	9086	19
42	3535	9354	3697	9291	3859	9225	4019	9157	4179	9085	18
43	3537	9353	3700	9290	3862	9224	4022	9155	4181	9084	17
44	3540	9352	3703	9289	3864	9223	4025	9154	4184	9083	16
45	3543	9351	3706	9288	3867	9222	4027	9153	4187	9081	15
46	3546	9350	3708	9287	3870	9221	4030	9152	4189	9080	14
47	3548	9349	3711	9286	3872	9220	4033	9151	4192	9079	13
48	3551	9348	3714	9285	3875	9219	4035	9150	4195	9078	12
49	3554	9347	3716	9284	3878	9218	4038	9148	4197	9077	11
50	3557	9346	3719	9283	3881	9216	4041	9147	4200	9075	10
51	3559	9345	3722	9282	3883	9215	4043	9146	4202	9074	9
52	3562	9344	3724	9281	3886	9214	4046	9145	4205	9073	8
53	3565	9343	3727	9279	3889	9213	4049	9144	4208	9072	7
54	3567	9342	3730	9278	3891	9212	4051	9143	4210	9070	6
55	3570	9341	3733	9277	3894	9211	4054	9141	4213	9069	5
56	3573	9340	3735	9276	3897	9210	4057	9140	4216	9068	4
57	3576	9339	3738	9275	3899	9208	4059	9139	4218	9067	3
58	3578	9338	3741	9274	3902	9207	4062	9138	4221	9066	2
59	3581	9337	3743	9273	3905	9206	4065	9137	4224	9064	1
60	3584	9336	3746	9272	3907	9205	4067	9135	4226	9063	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	69°		68°		67°		66°		65°		′

′	25°		26°		27°		28°		29°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	4226	9063	4384	8988	4540	8910	4695	8829	4848	8746	**60**
1	4229	9062	4386	8987	4542	8909	4697	8828	4851	8745	59
2	4231	9061	4389	8985	4545	8907	4700	8827	4853	8743	58
3	4234	9059	4392	8984	4548	8906	4702	8825	4856	8742	57
4	4237	9058	4394	8983	4550	8905	4705	8824	4858	8741	56
5	4239	9057	4397	8982	4553	8903	4708	8823	4861	8739	55
6	4242	9056	4399	8980	4555	8902	4710	8821	4863	8738	54
7	4245	9054	4402	8979	4558	8901	4713	8820	4866	8736	53
8	4247	9053	4405	8978	4561	8899	4715	8819	4868	8735	52
9	4250	9052	4407	8976	4563	8898	4718	8817	4871	8733	51
10	4253	9051	4410	8975	4566	8897	4720	8816	4874	8732	**50**
11	4255	9050	4412	8974	4568	8895	4723	8814	4876	8731	49
12	4258	9048	4415	8973	4571	8894	4726	8813	4879	8729	48
13	4260	9047	4418	8971	4574	8893	4728	8812	4881	8728	47
14	4263	9046	4420	8970	4576	8892	4731	8810	4884	8726	46
15	4266	9045	4423	8969	4579	8890	4733	8809	4886	8725	45
16	4268	9043	4425	8967	4581	8889	~~4736~~	~~8808~~	4889	8724	44
17	4271	9042	4428	8966	4584	8888	4738	8806	4891	8722	43
18	4274	9041	4431	8965	4586	8886	4741	8805	4894	8721	42
19	4276	9040	4433	8964	4589	8885	4743	8803	4896	8719	41
20	4279	9038	4436	8962	4592	8884	4746	8802	4899	8718	**40**
21	4281	9037	4439	8961	4594	8882	4749	8801	4901	8716	39
22	4284	9036	4441	8960	4597	8881	4751	8799	4904	8715	38
23	4287	9035	4444	8958	4599	8879	4754	8798	4907	8714	37
24	4289	9033	4446	8957	4602	8878	4756	8796	4909	8712	36
25	4292	9032	4449	8956	4605	8877	4759	8795	4912	8711	35
26	4295	9031	4452	8955	4607	8875	4761	8794	4914	8709	34
27	4297	9030	4454	8953	4610	8874	4764	8792	4917	8708	33
28	4300	9028	4457	8952	4612	8873	4766	8791	4919	8706	32
29	4302	9027	4459	8951	4615	8871	4769	8790	4922	8705	31
30	4305	9026	4462	8949	4617	8870	4772	8788	4924	8704	**30**
31	4308	9025	4465	8948	4620	8869	4774	8787	4927	8702	29
32	4310	9023	4467	8947	4623	8867	4777	8785	4929	8701	28
33	4313	9022	4470	8945	4625	8866	4779	8784	4932	8699	27
34	4316	9021	4472	8944	4628	8865	4782	8783	4934	8698	26
35	4318	9020	4475	8943	4630	8863	4784	8781	4937	8696	25
36	4321	9018	4478	8942	4633	8862	4787	8780	4939	8695	24
37	4323	9017	4480	8940	4636	8861	4789	8778	4942	8694	23
38	4326	9016	4483	8939	4638	8859	4792	8777	4944	8692	22
39	4329	9015	4485	8938	4641	8858	4795	8776	4947	8691	21
40	4331	9013	4488	8936	4643	8857	4797	8774	4950	8689	**20**
41	4334	9012	4491	8935	4646	8855	4800	8773	4952	8688	19
42	4337	9011	4493	8934	4648	8854	4802	8771	4955	8686	18
43	4339	9010	4496	8932	4651	8853	4805	8770	4957	8685	17
44	4342	9008	4498	8931	4654	8851	4807	8769	4960	8683	16
45	4344	9007	4501	8930	4656	8850	4810	8767	4962	8682	15
46	4347	9006	4504	8928	4659	8849	4812	8766	4965	8681	14
47	4350	9004	4506	8927	4661	8847	4815	8764	4967	8679	13
48	4352	9003	4509	8926	4664	8846	4818	8763	4970	8678	12
49	4355	9002	4511	8925	4666	8844	4820	8762	4972	8676	11
50	4358	9001	4514	8923	4669	8843	4823	8760	4975	8675	**10**
51	4360	8999	4517	8922	4672	8842	4825	8759	4977	8673	9
52	4363	8998	4519	8921	4674	8840	4828	8757	4980	8672	8
53	4365	8997	4522	8919	4677	8839	4830	8756	4982	8670	7
54	4368	8996	4524	8918	4679	8838	4833	8755	4985	8669	6
55	4371	8994	4527	8917	4682	8836	4835	8753	4987	8668	5
56	4373	8993	4530	8915	4684	8835	4838	8752	4990	8666	4
57	4376	8992	4532	8914	4687	8834	4840	8750	4992	8665	3
58	4378	8990	4535	8913	4690	8832	4843	8749	4995	8663	2
59	4381	8989	4537	8911	4692	8831	4846	8748	4997	8662	1
60	4384	8988	4540	8910	4695	8829	4848	8746	5000	8660	**0**
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	5000	8660	5150	8572	5299	8480	5446	8387	5592	8290	**60**
1	5003	8659	5153	8570	5302	8479	5449	8385	5594	8289	59
2	5005	8657	5155	8569	5304	8477	5451	8384	5597	8287	58
3	5008	8656	5158	8567	5307	8476	5454	8382	5599	8285	57
4	5010	8654	5160	8566	5309	8474	5456	8380	5602	8284	56
5	5013	8653	5163	8564	5312	8473	5459	8379	5604	8282	55
6	5015	8652	5165	8563	5314	8471	5461	8377	5606	8281	54
7	5018	8650	5168	8561	5316	8470	5463	8376	5609	8279	53
8	5020	8649	5170	8560	5319	8468	5466	8374	5611	8277	52
9	5023	8647	5173	8558	5321	8467	5468	8372	5614	8276	51
10	5025	8646	5175	8557	5324	8465	5471	8371	5616	8274	**50**
11	5028	8644	5178	8555	5326	8463	5473	8369	5618	8272	49
12	5030	8643	5180	8554	5329	8462	5476	8368	5621	8271	48
13	5033	8641	5183	8552	5331	8460	5478	8366	5623	8269	47
14	5035	8640	5185	8551	5334	8459	5480	8364	5626	8268	46
15	5038	8638	5188	8549	5336	8457	5583	8363	5628	8266	45
16	5040	8637	5190	8548	5339	8456	5485	8361	5630	8264	44
17	5043	8635	5193	8546	5341	8454	5488	8360	5633	8263	43
18	5045	8634	5195	8545	5344	8453	5490	8358	5635	8261	42
19	5048	8632	5198	8543	5346	8451	5493	8356	5638	8259	41
20	5050	8631	5200	8542	5348	8450	5495	8355	5640	8258	**40**
21	5053	8630	5203	8540	5351	8448	5498	8353	5642	8256	39
22	5055	8628	5205	8539	5353	8446	5500	8352	5645	8254	38
23	5058	8627	5208	8537	5356	8445	5502	8350	5647	8253	37
24	5060	8625	5210	8536	5358	8443	5505	8348	5650	8251	36
25	5063	8624	5213	8534	5361	8442	5507	8347	5652	8249	35
26	5065	8622	5215	8532	5363	8440	5510	8345	5654	8248	34
27	5068	8621	5218	8531	5366	8439	5512	8344	5657	8246	33
28	5070	8619	5220	8529	5368	8437	5515	8342	5659	8245	32
29	5073	8618	5223	8528	5371	8435	5517	8340	5662	8243	31
30	5075	8616	5225	8526	5373	8434	5519	8339	5664	8241	**30**
31	5078	8615	5227	8525	5375	8432	5522	8337	5666	8240	29
32	5080	8613	5230	8523	5378	8431	5524	8336	5669	8238	28
33	5083	8612	5232	8522	5380	8429	5527	8334	5671	8236	27
34	5085	8610	5235	8520	5383	8428	5529	8332	5674	8235	26
35	5088	8609	5237	8519	5385	8426	5531	8331	5676	8233	25
36	5090	8607	5240	8517	5388	8425	5534	8329	5678	8231	24
37	5093	8606	5242	8516	5390	8423	5536	8328	5681	8230	23
38	5095	8604	5245	8514	5393	8421	5539	8326	5683	8228	22
39	5098	8603	5247	8513	5395	8420	5541	8324	5686	8226	21
40	5100	8601	5250	8511	5398	8418	5544	8323	5688	8225	**20**
41	5103	8600	5252	8510	5400	8417	5546	8321	5690	8223	19
42	5105	8599	5255	8508	5402	8415	5548	8320	5693	8221	18
43	5108	8597	5257	8507	5405	8414	5551	8318	5695	8220	17
44	5110	8596	5260	8505	5407	8412	5553	8316	5698	8218	16
45	5113	8594	5262	8504	5410	8410	5556	8315	5700	8216	15
46	5115	8593	5265	8502	5412	8409	5558	8313	5702	8215	14
47	5118	8591	5267	8500	5415	8407	5561	8311	5705	8213	13
48	5120	8590	5270	8499	5417	8406	5563	8310	5707	8211	12
49	5123	8588	5272	8497	5420	8404	5565	8308	5710	8210	11
50	5125	8587	5275	8496	5422	8403	5568	8307	5712	8208	**10**
51	5128	8585	5277	8494	5424	8401	5570	8305	5714	8207	9
52	5130	8584	5279	8493	5427	8399	5573	8303	5717	8205	8
53	5133	8582	5282	8491	5429	8398	5575	8302	5719	8203	7
54	5135	8581	5284	8490	5432	8396	5577	8300	5721	8202	6
55	5138	8579	5287	8488	5434	8395	5580	8299	5724	8200	5
56	5140	8578	5289	8487	5437	8393	5582	8297	5726	8198	4
57	5143	8576	5292	8485	5439	8391	5585	8295	5729	8197	3
58	5145	8575	5294	8484	5442	8390	5587	8294	5731	8195	2
59	5148	8573	5297	8482	5444	8388	5590	8292	5733	8193	1
60	5150	8572	5299	8480	5446	8387	5592	8290	5736	8192	**0**
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	59°		58°		57°		56°		55°		′

′	35°		36°		37°		38°		39°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	5736	8192	5878	8090	6018	7986	6157	7880	6293	7771	60
1	5738	8190	5880	8088	6020	7985	6159	7878	6295	7770	59
2	5741	8188	5883	8087	6023	7983	6161	7877	6298	7768	58
3	5743	8187	5885	8085	6025	7981	6163	7875	6300	7766	57
4	5745	8185	5887	8083	6027	7979	6166	7873	6302	7764	56
5	5748	8183	5890	8082	6030	7978	6168	7871	6305	7762	55
6	5750	8181	5892	8080	6032	7976	6170	7869	6307	7760	54
7	5752	8180	5894	8078	6034	7974	6173	7868	6309	7759	53
8	5755	8178	5897	8076	6037	7972	6175	7866	6311	7757	52
9	5757	8176	5899	8075	6039	7971	6177	7864	6314	7755	51
10	5760	8175	5901	8073	6041	7969	6180	7862	6316	7753	50
11	5762	8173	5904	8071	6044	7967	6182	7860	6318	7751	49
12	5764	8171	5906	8070	6046	7965	6184	7859	6320	7749	48
13	5767	8170	5908	8068	6048	7964	6186	7857	6323	7748	47
14	5769	8168	5911	8066	6051	7962	6189	7855	6325	7746	46
15	5771	8166	5913	8064	6053	7960	6191	7853	6327	7744	45
16	5774	8165	5915	8063	6055	7958	6193	7851	6329	7742	44
17	5776	8163	5918	8061	6058	7956	6196	7850	6332	7740	43
18	5779	8161	5920	8059	6060	7955	6198	7848	6334	7738	42
19	5781	8160	5922	8058	6062	7953	6200	7846	6336	7737	41
20	5783	8158	5925	8056	6065	7951	6202	7844	6338	7735	40
21	5786	8156	5927	8054	6067	7950	6205	7842	6341	7733	39
22	5788	8155	5930	8052	6069	7948	6207	7841	6343	7731	38
23	5790	8153	5932	8051	6071	7946	6209	7839	6345	7729	37
24	5793	8151	5934	8049	6074	7944	6211	7837	6347	7727	36
25	5795	8150	5937	8047	6076	7942	6214	7835	6350	7725	35
26	5798	8148	5939	8045	6078	7941	6216	7833	6352	7724	34
27	5800	8146	5941	8044	6081	7939	6218	7832	6354	7722	33
28	5802	8145	5944	8042	6083	7937	6221	7830	6356	7720	32
29	5805	8143	5946	8040	6085	7935	6223	7828	6359	7718	31
30	5807	8141	5948	8039	6088	7934	6225	7826	6261	7716	30
31	5809	8139	5951	8037	6090	7932	6227	7824	6363	7714	29
32	5812	8138	5953	8035	6092	7930	6230	7822	6365	7713	28
33	5814	8136	5955	8033	6095	7928	6232	7821	6368	7711	27
34	5816	8134	5958	8032	6097	7926	6234	7819	6370	7709	26
35	5819	8133	5960	8030	6099	7925	6237	7817	6372	7707	25
36	5821	8131	5962	8028	6101	7923	6239	7815	6374	7705	24
37	5824	8129	5965	8026	6104	7921	6241	7813	6376	7703	23
38	5826	8128	5967	8025	6106	7919	6243	7812	6379	7701	22
39	5828	8126	5969	8023	6108	7918	6246	7810	6381	7700	21
40	5831	8124	5972	8021	6111	7916	6248	7808	6383	7698	20
41	5833	8123	5974	8020	6113	7914	6250	7806	6385	7696	19
42	5835	8121	5976	8018	6115	7912	6252	7804	6388	7694	18
43	5838	8119	5979	8016	6118	7910	6255	7802	6390	7692	17
44	5840	8117	5981	8014	6120	7909	6257	7801	6392	7690	16
45	5842	8116	5983	8013	6122	7907	6259	7799	6394	7688	15
46	5845	8114	5986	8011	6124	7905	6262	7797	6397	7687	14
47	5847	8112	5988	8009	6127	7903	6264	7795	6399	7685	13
48	5850	8111	5990	8007	6129	7902	6266	7793	6401	7683	12
49	5852	8109	5993	8006	6131	7900	6268	7792	6403	7681	11
50	5854	8107	5995	8004	6134	7898	6271	7790	6406	7679	10
51	5857	8106	5997	8002	6136	7896	6273	7788	6408	7677	9
52	5859	8104	6000	8000	6138	7894	6275	7786	6410	7675	8
53	5861	8102	6002	7999	6141	7893	6277	7784	6412	7674	7
54	5864	8100	6004	7997	6143	7891	6280	7782	6414	7672	6
55	5866	8099	6007	7995	6145	7889	6282	7781	6417	7670	5
56	5868	8097	6009	7993	6147	7887	6284	7779	6419	7668	4
57	5871	8095	6011	7992	6150	7885	6286	7777	6421	7666	3
58	5873	8094	6014	7990	6152	7884	6289	7775	6423	7664	2
59	5875	8092	6016	7988	6154	7882	6291	7773	6426	7662	1
60	5878	8090	6018	7986	6157	7880	6293	7771	6428	7660	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	54°		53°		52°		51°		50°		′

′	40°		41°		42°		43°		44°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	6428	7660	6561	7547	6691	7431	6820	7314	6947	7193	60
1	6430	7659	6563	7545	6693	7430	6822	7312	6949	7191	59
2	6432	7657	6565	7543	6696	7428	6824	7310	6951	7189	58
3	6435	7655	6567	7541	6698	7426	6826	7308	6953	7187	57
4	6437	7653	6569	7539	6700	7424	6828	7306	6955	7185	56
5	6439	7651	6572	7538	6702	7422	6831	7304	6957	7183	55
6	6441	7649	6574	7536	6704	7420	6833	7302	6959	7181	54
7	6443	7647	6576	7534	6706	7418	6835	7300	6961	7179	53
8	6446	7645	6578	7532	6709	7416	6837	7298	6963	7177	52
9	6448	7644	6580	7530	6711	7414	6839	7296	6965	7175	51
10	6450	7642	6583	7528	6713	7412	6841	7294	6967	7173	50
11	6452	7640	6585	7526	6715	7410	6843	7292	6970	7171	49
12	6455	7638	6587	7524	6717	7408	6845	7290	6972	7169	48
13	6457	7636	6589	7522	6719	7406	6848	7288	6974	7167	47
14	6459	7634	6591	7520	6722	7404	6850	7286	6976	7165	46
15	6461	7632	6593	7518	6724	7402	6852	7284	6978	7163	45
16	6463	7630	6596	7516	6726	7400	6854	7282	6980	7161	44
17	6466	7629	6598	7515	6728	7398	6856	7280	6982	7159	43
18	6468	7627	6600	7513	6730	7396	6858	7278	6984	7157	42
19	6470	7625	6602	7511	6732	7394	6860	7276	6986	7155	41
20	6472	7623	6604	7509	6734	7392	6862	7274	6988	7153	40
21	6475	7621	6607	7507	6737	7390	6865	7272	6990	7151	39
22	6477	7619	6609	7505	6739	7388	6867	7270	6992	7149	38
23	6479	7617	6611	7503	6741	7387	6869	7268	6995	7147	37
24	6481	7615	6613	7501	6743	7385	6871	7266	6997	7145	36
25	6483	7613	6615	7499	6745	7383	6873	7264	6999	7143	35
26	6486	7612	6617	7497	6747	7381	6875	7262	7001	7141	34
27	6488	7610	6620	7495	6749	7379	6877	7260	7003	7139	33
28	6490	7608	6622	7493	6752	7377	6879	7258	7005	7137	32
29	6492	7606	6624	7491	6754	7375	6881	7256	7007	7135	31
30	6494	7604	6626	7490	6756	7373	6884	7254	7009	7133	30
31	6497	7602	6628	7488	6758	7371	6886	7252	7011	7130	29
32	6499	7600	6631	7486	6760	7369	6888	7250	7013	7128	28
33	6501	7598	6633	7484	6762	7367	6890	7248	7015	7126	27
34	6503	7596	6635	7482	6764	7365	6892	7246	7017	7124	26
35	6506	7595	6637	7480	6767	7363	6894	7244	7019	7122	25
36	6508	7593	6639	7478	6769	7361	6896	7242	7022	7120	24
37	6510	7591	6641	7476	6771	7359	6898	7240	7024	7118	23
38	6512	7589	6644	7474	6773	7357	6900	7238	7026	7116	22
39	6514	7587	6646	7472	6775	7355	6903	7236	7028	7114	21
40	6517	7585	6648	7470	6777	7353	6905	7234	7030	7112	20
41	6519	7583	6650	7468	6779	7351	6907	7232	7032	7110	19
42	6521	7581	6652	7466	6782	7349	6909	7230	7034	7108	18
43	6523	7579	6654	7464	6784	7347	6911	7228	7036	7106	17
44	6525	7578	6657	7463	6786	7345	6913	7226	7038	7104	16
45	6528	7576	6659	7461	6788	7343	6915	7224	7040	7102	15
46	6530	7574	6661	7459	6790	7341	6917	7222	7042	7100	14
47	6532	7572	6663	7457	6792	7339	6919	7220	7044	7098	13
48	6534	7570	6665	7455	6794	7337	6921	7218	7046	7096	12
49	6536	7568	6667	7453	6797	7335	6924	7216	7048	7094	11
50	6539	7566	6670	7451	6799	7333	6926	7214	7050	7092	10
51	6541	7564	6672	7449	6801	7331	6928	7212	7053	7090	9
52	6543	7562	6674	7447	6803	7329	6930	7210	7055	7088	8
53	6545	7560	6676	7445	6805	7327	6932	7208	7057	7085	7
54	6547	7559	6678	7443	6807	7325	6934	7206	7059	7083	6
55	6550	7557	6680	7441	6809	7323	6936	7203	7061	7081	5
56	6552	7555	6683	7439	6811	7321	6938	7201	7063	7079	4
57	6554	7553	6685	7437	6814	7319	6940	7199	7065	7077	3
58	6556	7551	6687	7435	6816	7318	6942	7197	7067	7075	2
59	6558	7549	6689	7433	6818	7316	6944	7195	7069	7073	1
60	6561	7547	6691	7431	6820	7314	6947	7193	7071	7071	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	49°		48°		47°		46°		45°		′

TABLE IV

NATURAL TANGENTS
AND
COTANGENTS

′	0°		1°		2°		3°		4°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	0000	Infinity	0175	57.2900	0349	28.6363	0524	19.0811	0699	14.3007	**60**
1	0003	3437.75	0177	56.3506	0352	28.3994	0527	18.9755	0702	14.2411	59
2	0006	1718.87	0180	55.4415	0355	28.1664	0530	18.8711	0705	14.1821	58
3	0009	1145.92	0183	54.5613	0358	27.9372	0533	18.7678	0708	14.1235	57
4	0012	859.436	0186	53.7086	0361	27.7117	0536	18.6656	0711	14.0655	56
5	0015	687.549	0189	52.8821	0364	27.4899	0539	18.5645	0714	14.0079	55
6	0017	572.957	0192	52.0807	0367	27.2715	0542	18.4645	0717	13.9507	54
7	0020	491.106	0195	51.3032	0370	27.0566	0544	18.3655	0720	13.8940	53
8	0023	429.718	0198	50.5485	0373	26.8450	0547	18.2677	0723	13.8378	52
9	0026	381.971	0201	49.8157	0375	26.6367	0550	18.1708	0726	13.7821	51
10	0029	343.774	0204	49.1039	0378	26.4316	0553	18.0750	0729	13.7267	**50**
11	0032	312.521	0207	48.4121	0381	26.2296	0556	17.9802	0731	13.6719	49
12	0035	286.478	0209	47.7395	0384	26.0307	0559	17.8863	0734	13.6174	48
13	0038	264.441	0212	47.0853	0387	25.8348	0562	17.7934	0737	13.5634	47
14	0041	245.552	0215	46.4489	0390	25.6418	0565	17.7015	0740	13.5098	46
15	0044	229.182	0218	45.8294	0393	25.4517	0568	17.6106	0743	13.4566	45
16	0047	214.858	0221	45.2261	0396	25.2644	0571	17.5205	0746	13.4039	44
17	0049	202.219	0224	44.6386	0399	25.0798	0574	17.4314	0749	13.3515	43
18	0052	190.984	0227	44.0661	0402	24.8978	0577	17.3432	0752	13.2996	42
19	0055	180.932	0230	43.5081	0405	24.7185	0580	17.2558	0755	13.2480	41
20	0058	171.885	0233	42.9641	0407	24.5418	0582	17.1693	0758	13.1969	**40**
21	0061	163.700	0236	42.4335	0410	24.3675	0585	17.0837	0761	13.1461	39
22	0064	156.259	0239	41.9158	0413	24.1957	0588	16.9990	0764	13.0958	38
23	0067	149.465	0241	41.4106	0416	24.0263	0591	16.9150	0767	13.0458	37
24	0070	143.237	0244	40.9174	0419	23.8593	0594	16.8319	0769	12.9962	36
25	0073	137.507	0247	40.4358	0422	23.6945	0597	16.7496	0772	12.9469	35
26	0076	132.219	0250	39.9655	0425	23.5321	0600	16.6681	0775	12.8981	34
27	0079	127.321	0253	39.5059	0428	23.3718	0603	16.5874	0778	12.8496	33
28	0081	122.774	0256	39.0568	0431	23.2137	0606	16.5075	0781	12.8014	32
29	0084	118.540	0259	38.6177	0434	23.0577	0609	16.4283	0784	12.7536	31
30	0087	114.589	0262	38.1885	0437	22.9038	0612	16.3499	0787	12.7062	**30**
31	0090	110.892	0265	37.7686	0440	22.7519	0615	16.2722	0790	12.6591	29
32	0093	107.426	0268	37.3579	0442	22.6020	0617	16.1952	0793	12.6124	28
33	0096	104.171	0271	36.9560	0445	22.4541	0620	16.1190	0796	12.5660	27
34	0099	101.107	0274	36.5627	0448	22.3081	0623	16.0435	0799	12.5199	26
35	0102	98.2179	0276	36.1776	0451	22.1640	0626	15.9687	0802	12.4742	25
36	0105	95.4895	0279	35.8006	0454	22.0217	0629	15.8945	0805	12.4288	24
37	0108	92.9085	0282	35.4313	0457	21.8813	0632	15.8211	0808	12.3838	23
38	0111	90.4633	0285	35.0695	0460	21.7426	0635	15.7483	0810	12.3390	22
39	0113	88.1436	0288	34.7151	0463	21.6056	0638	15.6762	0813	12.2946	21
40	0116	85.9398	0291	34.3678	0466	21.4704	0641	15.6048	0816	12.2505	**20**
41	0119	83.8435	0294	34.0273	0469	21.3369	0644	15.5340	0819	12.2067	19
42	0122	81.8470	0297	33.6935	0472	21.2049	0647	15.4638	0822	12.1632	18
43	0125	79.9434	0300	33.3662	0475	21.0747	0650	15.3943	0825	12.1201	17
44	0128	78.1263	0303	33.0452	0477	20.9460	0653	15.3254	0828	12.0772	16
45	0131	76.3900	0306	32.7303	0480	20.8188	0655	15.2571	0831	12.0346	15
46	0134	74.7292	0308	32.4213	0483	20.6932	0658	15.1893	0834	11.9923	14
47	0137	73.1390	0311	32.1181	0486	20.5691	0661	15.1222	0837	11.9504	13
48	0140	71.6151	0314	31.8205	0489	20.4465	0664	15.0557	0840	11.9087	12
49	0143	70.1533	0317	31.5284	0492	20.3253	0667	14.9898	0843	11.8673	11
50	0146	68.7501	0320	31.2416	0495	20.2056	0670	14.9244	0846	11.8262	**10**
51	0148	67.4019	0323	30.9599	0498	20.0872	0673	14.8596	0849	11.7853	9
52	0151	66.1055	0326	30.6833	0501	19.9702	0676	14.7954	0851	11.7448	8
53	0154	64.8580	0329	30.4116	0504	19.8546	0679	14.7317	0854	11.7045	7
54	0157	63.6567	0332	30.1446	0507	19.7403	0682	14.6685	0857	11.6645	6
55	0160	62.4992	0335	29.8823	0509	19.6273	0685	14.6059	0860	11.6248	**5**
56	0163	61.3829	0338	29.6245	0512	19.5156	0688	14.5438	0863	11.5853	**4**
57	0166	60.3058	0340	29.3711	0515	19.4051	0690	14.4823	0866	11.5461	**3**
58	0169	59.2659	0343	29.1220	0518	19.2959	0693	14.4212	0869	11.5072	**2**
59	0172	58.2612	0346	28.8771	0521	19.1879	0696	14.3607	0872	11.4685	**1**
60	0175	57.2900	0349	28.6363	0524	19.0811	0699	14.3007	0875	11.4301	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	0875	11.4301	1051	9.5144	1228	8.1443	1405	7.1154	1584	6.3138	60
1	0878	11.3919	1054	9.4878	1231	8.1248	1408	7.1004	1587	6.3019	59
2	0881	11.3540	1057	9.4614	1234	8.1054	1411	7.0855	1590	6.2901	58
3	0884	11.3163	1060	9.4352	1237	8.0860	1414	7.0706	1593	6.2783	57
4	0887	11.2789	1063	9.4090	1240	8.0667	1417	7.0558	1596	6.2666	56
5	0890	11.2417	1066	9.3831	1243	8.0476	1420	7.0410	1599	6.2549	55
6	0892	11.2048	1069	9.3572	1246	8.0285	1423	7.0264	1602	6.2432	54
7	0895	11.1681	1072	9.3315	1249	8.0095	1426	7.0117	1605	6.2316	53
8	0898	11.1316	1075	9.3060	1251	7.9906	1429	6.9972	1608	6.2200	52
9	0901	11.0954	1078	9.2806	1254	7.9718	1432	6.9827	1611	6.2085	51
10	0904	11.0594	1080	9.2553	1257	7.9530	1435	6.9682	1614	6.1970	50
11	0907	11.0237	1083	9.2302	1260	7.9344	1438	6.9538	1617	6.1856	49
12	0910	10.9882	1086	9.2052	1263	7.9158	1441	6.9395	1620	6.1742	48
13	0913	10.9529	1089	9.1803	1266	7.8973	1444	6.9252	1623	6.1628	47
14	0916	10.9178	1092	9.1555	1269	7.8789	1447	6.9110	1626	6.1515	46
15	0919	10.8829	1095	9.1309	1272	7.8606	1450	6.8969	1629	6.1402	45
16	0922	10.8483	1098	9.1065	1275	7.8424	1453	6.8828	1632	6.1290	44
17	0925	10.8139	1101	9.0821	1278	7.8243	1456	6.8687	1635	6.1178	43
18	0928	10.7797	1104	9.0579	1281	7.8062	1459	6.8548	1638	6.1066	42
19	0931	10.7457	1107	9.0338	1284	7.7883	1462	6.8408	1641	6.0955	41
20	0934	10.7119	1110	9.0098	1287	7.7704	1465	6.8269	1644	6.0844	40
21	0936	10.6783	1113	8.9860	1290	7.7525	1468	6.8131	1647	6.0734	39
22	0939	10.6450	1116	8.9623	1293	7.7348	1471	6.7994	1650	6.0624	38
23	0942	10.6118	1119	8.9387	1296	7.7171	1474	6.7856	1653	6.0514	37
24	0945	10.5789	1122	8.9152	1299	7.6996	1477	6.7720	1655	6.0405	36
25	0948	10.5462	1125	8.8919	1302	7.6821	1480	6.7584	1658	6.0296	35
26	0951	10.5136	1128	8.8686	1305	7.6647	1483	6.7448	1661	6.0188	34
27	0954	10.4813	1131	8.8455	1308	7.6473	1486	6.7313	1664	6.0080	33
28	0957	10.4491	1134	8.8225	1311	7.6301	1489	6.7179	1667	5.9972	32
29	0960	10.4172	1136	8.7996	1314	7.6129	1492	6.7045	1670	5.9865	31
30	0963	10.3854	1139	8.7769	1317	7.5958	1495	6.6912	1673	5.9758	30
31	0966	10.3538	1142	8.7542	1319	7.5787	1497	6.6779	1676	5.9651	29
32	0969	10.3224	1145	8.7317	1322	7.5618	1500	6.6646	1679	5.9545	28
33	0972	10.2913	1148	8.7093	1325	7.5449	1503	6.6514	1682	5.9439	27
34	0975	10.2602	1151	8.6870	1328	7.5281	1506	6.6383	1685	5.9333	26
35	0978	10.2294	1154	8.6648	1331	7.5113	1509	6.6252	1688	5.9228	25
36	0981	10.1988	1157	8.6427	1334	7.4947	1512	6.6122	1691	5.9124	24
37	0983	10.1683	1160	8.6208	1337	7.4781	1515	6.5992	1694	5.9019	23
38	0986	10.1381	1163	8.5989	1340	7.4615	1518	6.5863	1697	5.8915	22
39	0989	10.1080	1166	8.5772	1343	7.4451	1521	6.5734	1700	5.8811	21
40	0992	10.0780	1169	8.5555	1346	7.4287	1524	6.5606	1703	5.8708	20
41	0995	10.0483	1172	8.5340	1349	7.4124	1527	6.5478	1706	5.8605	19
42	0998	10.0187	1175	8.5126	1352	7.3962	1530	6.5350	1709	5.8502	18
43	1001	9.9893	1178	8.4913	1355	7.3800	1533	6.5223	1712	5.8400	17
44	1004	9.9601	1181	8.4701	1358	7.3639	1536	6.5097	1715	5.8298	16
45	1007	9.9310	1184	8.4490	1361	7.3479	1539	6.4971	1718	5.8197	15
46	1010	9.9021	1187	8.4280	1364	7.3319	1542	6.4846	1721	5.8095	14
47	1013	9.8734	1189	8.4071	1367	7.3160	1545	6.4721	1724	5.7994	13
48	1016	9.8448	1192	8.3863	1370	7.3002	1548	6.4596	1727	5.7894	12
49	1019	9.8164	1195	8.3656	1373	7.2844	1551	6.4472	1730	5.7794	11
50	1022	9.7882	1198	8.3450	1376	7.2687	1554	6.4348	1733	5.7694	10
51	1025	9.7601	1201	8.3245	1379	7.2531	1557	6.4225	1736	5.7594	9
52	1028	9.7322	1204	8.3041	1382	7.2375	1560	6.4103	1739	5.7495	8
53	1030	9.7044	1207	8.2838	1385	7.2220	1563	6.3980	1742	5.7396	7
54	1033	9.6768	1210	8.2636	1388	7.2066	1566	6.3859	1745	5.7297	6
55	1036	9.6499	1213	8.2434	1391	7.1912	1569	6.3737	1748	5.7199	5
56	1039	9.6220	1216	8.2234	1394	7.1759	1572	6.3617	1751	5.7101	4
57	1042	9.5949	1219	8.2035	1397	7.1607	1575	6.3496	1754	5.7004	3
58	1045	9.5679	1222	8.1837	1399	7.1455	1578	6.3376	1757	5.6906	2
59	1048	9.5411	1225	8.1640	1402	7.1304	1581	6.3257	1760	5.6809	1
60	1051	9.5144	1228	8.1443	1405	7.1154	1584	6.3138	1763	5.6713	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
	84°		83°		82°		81°		80°		′

′	10°		11°		12°		13°		14°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	1763	5.6713	1944	5.1446	2126	4.7046	2309	4.3315	2493	4.0108	60
1	1766	5.6617	1947	5.1366	2129	4.6979	2312	4.3257	2496	4.0058	59
2	1769	5.6521	1950	5.1286	2132	4.6912	2315	4.3200	2499	4.0009	58
3	1772	5.6425	1953	5.1207	2135	4.6845	2318	4.3143	2503	3.9959	57
4	1775	5.6330	1956	5.1128	2138	4.6779	2321	4.3086	2506	3.9910	56
5	1778	5.6234	1959	5.1049	2141	4.6712	2324	4.3029	2509	3.9861	55
6	1781	5.6140	1962	5.0970	2144	4.6646	2327	4.2972	2512	3.9812	54
7	1784	5.6045	1965	5.0892	2147	4.6580	2330	4.2916	2515	3.9763	53
8	1787	5.5951	1968	5.0814	2150	4.6514	2333	4.2859	2518	3.9714	52
9	1790	5.5857	1971	5.0736	2153	4.6448	2336	4.2803	2521	3.9665	51
10	1793	5.5764	1974	5.0658	2156	4.6382	2339	4.2747	2524	3.9617	50
11	1796	5.5671	1977	5.0581	2159	4.6317	2342	4.2691	2527	3.9568	49
12	1799	5.5578	1980	5.0504	2162	4.6252	2345	4.2635	2530	3.9520	48
13	1802	5.5485	1983	5.0427	2165	4.6187	2349	4.2580	2533	3.9471	47
14	1805	5.5393	1986	5.0350	2168	4.6122	2352	4.2524	2537	3.9423	46
15	1808	5.5301	1989	5.0273	2171	4.6057	2355	4.2468	2540	3.9375	45
16	1811	5.5209	1992	5.0197	2174	4.5993	2358	4.2413	2543	3.9327	44
17	1814	5.5118	1995	5.0121	2177	4.5928	2361	4.2358	2546	3.9279	43
18	1817	5.5026	1998	5.0045	2180	4.5864	2364	4.2303	2549	3.9232	42
19	1820	5.4936	2001	4.9969	2183	4.5800	2367	4.2248	2552	3.9184	41
20	1823	5.4845	2004	4.9894	2186	4.5736	2370	4.2193	2555	3.9136	40
21	1826	5.4755	2007	4.9819	2189	4.5673	2373	4.2139	2558	3.9089	39
22	1829	5.4665	2010	4.9744	2193	4.5609	2376	4.2084	2561	3.9042	38
23	1832	5.4575	2013	4.9669	2196	4.5546	2379	4.2030	2564	3.8995	37
24	1835	5.4486	2016	4.9594	2199	4.5483	2382	4.1976	2568	3.8947	36
25	1838	5.4397	2019	4.9520	2202	4.5420	2385	4.1922	2571	3.8900	35
26	1841	5.4308	2022	4.9446	2205	4.5357	2388	4.1868	2574	3.8854	34
27	1844	5.4219	2025	4.9372	2208	4.5294	2392	4.1814	2577	3.8807	33
28	1847	5.4131	2028	4.9298	2211	4.5232	2395	4.1760	2580	3.8760	32
29	1850	5.4043	2031	4.9225	2214	4.5169	2398	4.1706	2583	3.8714	31
30	1853	5.3955	2035	4.9152	2217	4.5107	2401	4.1653	2586	3.8667	30
31	1856	5.3868	2038	4.9078	2220	4.5045	2404	4.1600	2589	3.8621	29
32	1859	5.3781	2041	4.9006	2223	4.4983	2407	4.1547	2592	3.8575	28
33	1862	5.3694	2044	4.8933	2226	4.4922	2410	4.1493	2595	3.8528	27
34	1865	5.3607	2047	4.8860	2229	4.4860	2413	4.1441	2599	3.8482	26
35	1868	5.3521	2050	4.8788	2232	4.4799	2416	4.1388	2602	3.8436	25
36	1871	5.3435	2053	4.8716	2235	4.4737	2419	4.1335	2605	3.8391	24
37	1874	5.3349	2056	4.8644	2238	4.4676	2422	4.1282	2608	3.8345	23
38	1877	5.3263	2059	4.8573	2241	4.4615	2425	4.1230	2611	3.8299	22
39	1880	5.3178	2062	4.8501	2244	4.4555	2428	4.1178	2614	3.8254	21
40	1883	5.3093	2065	4.8430	2247	4.4494	2432	4.1126	2617	3.8208	20
41	1887	5.3008	2068	4.8359	2251	4.4434	2435	4.1074	2620	3.8163	19
42	1890	5.2924	2071	4.8288	2254	4.4374	2438	4.1022	2623	3.8118	18
43	1893	5.2839	2074	4.8218	2257	4.4313	2441	4.0970	2627	3.8073	17
44	1896	5.2755	2077	4.8147	2260	4.4253	2444	4.0918	2630	3.8028	16
45	1899	5.2672	2080	4.8077	2263	4.4194	2447	4.0867	2633	3.7983	15
46	1902	5.2588	2083	4.8007	2266	4.4134	2450	4.0815	2636	3.7938	14
47	1905	5.2505	2086	4.7937	2269	4.4075	2453	4.0764	2639	3.7893	13
48	1908	5.2422	2089	4.7867	2272	4.4015	2456	4.0713	2642	3.7848	12
49	1911	5.2339	2092	4.7798	2275	4.3956	2459	4.0662	2645	3.7804	11
50	1914	5.2257	2095	4.7729	2278	4.3897	2462	4.0611	2648	3.7760	10
51	1917	5.2174	2098	4.7659	2281	4.3838	2465	4.0560	2651	3.7715	9
52	1920	5.2092	2101	4.7591	2284	4.3779	2469	4.0509	2655	3.7671	8
53	1923	5.2011	2104	4.7522	2287	4.3721	2472	4.0459	2658	3.7627	7
54	1926	5.1929	2107	4.7453	2290	4.3662	2475	4.0408	2661	3.7583	6
55	1929	5.1848	2110	4.7385	2293	4.3604	2478	4.0358	2664	3.7539	5
56	1932	5.1767	2113	4.7317	2296	4.3546	2481	4.0308	2667	3.7495	4
57	1935	5.1686	2116	4.7249	2299	4.3488	2484	4.0257	2670	3.7451	3
58	1938	5.1606	2119	4.7181	2303	4.3430	2487	4.0207	2673	3.7408	2
59	1941	5.1526	2123	4.7114	2306	4.3372	2490	4.0158	2676	3.7364	1
60	1944	5.1446	2126	4.7046	2309	4.3315	2493	4.0108	2679	3.7321	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	79°		78°		77°		76°		75°		′

′	15°		16°		17°		18°		19°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	2679	3.7321	2867	3.4874	3057	3.2709	3249	3.0777	3443	2.9042	60
1	2683	3.7277	2871	3.4836	3060	3.2675	3252	3.0746	3447	2.9015	59
2	2686	3.7234	2874	3.4798	3064	3.2641	3256	3.0716	3450	2.8987	58
3	2689	2.7191	2877	3.4760	3067	3.2607	3259	3.0686	3453	2.8960	57
4	2692	3.7148	2880	3.4722	3070	3.2573	3262	3.0655	3456	2.8933	56
5	2695	3.7105	2883	3.4684	3073	3.2539	3265	3.0625	3460	2.8905	55
6	2698	3.7062	2886	3.4646	3076	3.2506	3269	3.0595	3463	2.8878	54
7	2701	2.7019	2890	3.4608	3080	3.2472	3272	3.0565	3466	2.8851	53
8	2704	3.6976	2893	3.4570	3083	3.2438	3275	3.0535	3469	2.8824	52
9	2708	3.6933	2896	3.4533	3086	3.2405	3278	3.0505	3473	2.8797	51
10	2711	3.6891	2899	3.4495	3089	3.2371	3281	3.0475	3476	2.8770	50
11	2714	3.6848	2902	3.4458	3092	3.2338	3285	3.0445	3479	2.8743	49
12	2717	3.6806	2905	3.4420	3096	3.2305	3288	3.0415	3482	2.8716	48
13	2720	3.6764	2908	3.4383	3099	3.2272	3291	3.0385	3486	2.8689	47
14	2723	3.6722	2912	3.4346	3102	3.2238	3294	3.0356	3489	2.8662	46
15	2726	3.6680	2915	3.4308	3105	3.2205	3298	3.0326	3492	2.8636	45
16	2729	3.6638	2918	3.4271	3108	3.2172	3301	3.0296	3495	2.8609	44
17	2733	3.6596	2921	3.4234	3111	3.2139	3304	3.0267	3499	2.8582	43
18	2736	3.6554	2924	3.4197	3115	3.2106	3307	3.0237	3502	2.8556	42
19	2739	3.6512	2927	3.4160	3118	3.2073	3310	3.0208	3505	2.8529	41
20	2742	3.6470	2931	3.4124	3121	3.2041	3314	3.0178	3508	2.8502	40
21	2745	3.6429	2934	3.4087	3124	3.2008	3317	3.0149	3512	2.8476	39
22	2748	3.6387	2937	3.4050	3127	3.1975	3320	3.0120	3515	2.8449	38
23	2751	3.6346	2940	3.4014	3131	3.1943	3323	3.0090	3518	2.8423	37
24	2754	3.6305	2943	3.3977	3134	3.1910	3327	3.0061	3522	2.8397	36
25	2758	3.6264	2946	3.3941	3137	3.1878	3330	3.0032	3525	2.8370	35
26	2761	3.6222	2949	3.3904	3140	3.1845	3333	3.0003	3528	2.8344	34
27	2764	2.6181	2953	3.3868	3143	3.1813	3336	2.9974	3531	2.8318	33
28	2767	3.6140	2956	3.3832	3147	3.1780	3339	2.9945	3535	2.8291	32
29	2770	3.6100	2959	3.3796	3150	3.1748	3343	2.9916	3538	2.8265	31
30	2773	3.6059	2962	3.3759	3153	3.1716	3346	2.9887	3541	2.8239	30
31	2776	3.6018	2965	3.3723	3156	3.1684	3349	2.9858	3544	2.8213	29
32	2780	3.5978	2968	3.3687	3159	3.1652	3352	2.9829	3548	2.8187	28
33	2783	3.5937	2972	3.3652	3163	3.1620	3356	2.9800	3551	2.8161	27
34	2786	3.5897	2975	3.3616	3166	3.1588	3359	2.9772	3554	2.8135	26
35	2789	3.5856	2978	3.3580	3169	3.1556	3362	2.9743	3558	2.8109	25
36	2792	3.5816	2981	3.3544	3172	3.1524	3365	2.9714	3561	2.8083	24
37	2795	3.5776	2984	3.3509	3175	3.1492	3369	2.9686	3564	2.8057	23
38	2798	3.5736	2987	3.3473	3179	3.1460	3372	2.9657	3567	2.8032	22
39	2801	3 5696	2991	3.3438	3182	3.1429	3375	2.9629	3571	2.8006	21
40	2805	3.5656	2994	3.3402	3185	3.1397	3378	2.9600	3574	2.7980	20
41	2808	3.5616	2997	3.3367	3188	3.1366	3382	2.9572	3577	2.7955	19
42	2811	3.5576	3000	3.3332	3191	3.1334	3385	2.9544	3581	2.7929	18
43	2814	3.5536	3003	3.3297	3195	3.1303	3388	2.9515	3584	2.7903	17
44	2817	3.5497	3006	3.3261	3198	3.1271	3391	2.9487	3587	2.7878	16
45	2820	3.5457	3010	3.3226	3201	3.1240	3395	2.9459	3590	2.7852	15
46	2823	3.5418	3013	3.3191	3204	3.1209	3398	2.9431	3594	2.7827	14
47	2827	3.5379	3016	3.3156	3207	3.1178	3401	2.9403	3597	2.7801	13
48	2830	3.5339	3019	3.3122	3211	3.1146	3404	2.9375	3600	2.7776	12
49	2833	3.5300	3022	3.3087	3214	3.1115	3408	2.9347	3604	2.7751	11
50	2836	3.5261	3026	3.3052	3217	3.1084	3411	2.9319	3607	2.7725	10
51	2839	3.5222	3029	3.3017	3220	3.1053	3414	2.9291	3610	2.7700	9
52	2842	3.5183	3032	3.2983	3223	3.1022	3417	2.9263	3613	2.7675	8
53	2845	3.5144	3035	3.2948	3227	3.0991	3421	2.9235	3617	2.7650	7
54	2849	3.5105	3038	3.2914	3230	3.0961	3424	2.9208	3620	2.7625	6
55	2852	3.5067	3041	3.2880	3233	3.0930	3427	2.9180	3623	2.7600	5
56	2855	3.5028	3045	3.2845	3236	3.0899	3430	2.9152	3627	2.7575	4
57	2858	3.4989	3048	3.2811	3240	3.0868	3434	2.9125	3630	2.7550	3
58	2861	3.4951	3051	3.2777	3243	3.0838	3437	2.9097	3633	2.7525	2
59	2864	3.4912	3054	3.2743	3246	3.0807	3440	2.9070	3636	2.7500	1
60	2867	3.4874	3057	3.2709	3249	3.0777	3443	2.9042	3640	2.7475	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	74°		73°		72°		71°		70°		′

′	20°		21°		22°		23°		24°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	3640	2.7475	3839	2.6051	4040	2.4751	4245	2.3559	4452	2.2460	60
1	3643	2.7450	3842	2.6028	4044	2.4730	4248	2.3539	4456	2.2443	59
2	3646	2.7425	3845	2.6006	4047	2.4709	4252	2.3520	4459	2.2425	58
3	3650	2.7400	3849	2.5983	4050	2.4689	4255	2.3501	4463	2.2408	57
4	3653	2.7376	3852	2.5961	4054	2.4668	4258	2.3483	4466	2.2390	56
5	3656	2.7351	3855	2.5938	4057	2.4648	4262	2.3464	4470	2.2373	55
6	3659	2.7326	3859	2.5916	4061	2.4627	4265	2.3445	4473	2.2355	54
7	3663	2.7302	3862	2.5893	4064	2.4606	4269	2.3426	4477	2.2338	53
8	3666	2.7277	3865	2.5871	4067	2.4586	4272	2.3407	4480	2.2320	52
9	3669	2.7253	3869	2.5848	4071	2.4566	4276	2.3388	4484	2.2303	51
10	3673	2.7228	3872	2.5826	4074	2.4545	4279	2.3369	4487	2.2286	50
11	3676	2.7204	3875	2.5804	4078	2.4525	4283	2.3351	4491	2.2268	49
12	3679	2.7179	3879	2.5782	4081	2.4504	4286	2.3332	4494	2.2251	48
13	3683	2.7155	3882	2.5759	4084	2.4484	4289	2.3313	4498	2.2234	47
14	3686	2.7130	3885	2.5737	4088	2.4464	4293	2.3294	4501	2.2216	46
15	3689	2.7106	3889	2.5715	4091	2.4443	4296	2.3276	4505	2.2199	45
16	3693	2.7082	3892	2.5693	4095	2.4423	4300	2.3257	4508	2.2182	44
17	3696	2.7058	3895	2.5671	4098	2.4403	4303	2.3238	4512	2.2165	43
18	3699	2.7034	3899	2.5649	4101	2.4383	4307	2.3220	4515	2.2148	42
19	3702	2.7009	3902	2.5627	4105	2.4362	4310	2.3201	4519	2.2130	41
20	3706	2.6985	3906	2.5605	4108	2.4342	4314	2.3183	4522	2.2113	40
21	3709	2.6961	3909	2.5583	4111	2.4322	4317	2.3164	4526	2.2096	39
22	3712	2.6937	3912	2.5561	4115	2.4302	4320	2.3146	4529	2.2079	38
23	3716	2.6913	3916	2.5539	4118	2.4282	4324	2.3127	4533	2.2062	37
24	3719	2.6889	3919	2.5517	4122	2.4262	4327	2.3109	4536	2.2045	36
25	3722	2.6865	3922	2.5495	4125	2.4242	4331	2.3090	4540	2.2028	35
26	3726	2.6841	3926	2.5473	4129	2.4222	4334	2.3072	4543	2.2011	34
27	3729	2.6818	3929	2.5452	4132	2.4202	4338	2.3053	4547	2.1994	33
28	3732	2.6794	3932	2.5430	4135	2.4182	4341	2.3035	4550	2.1977	32
29	3736	2.6770	3936	2.5408	4139	2.4162	4345	2.3017	4554	2.1960	31
30	3739	2.6746	3939	2.5386	4142	2.4142	4348	2.2998	4557	2.1943	30
31	3742	2.6723	3942	2.5365	4146	2.4122	4352	2.2980	4561	2.1926	29
32	3745	2.6699	3946	2.5343	4149	2.4102	4355	2.2962	4564	2.1909	28
33	3749	2.6675	3949	2.5322	4152	2.4083	4359	2.2944	4568	2.1892	27
34	3752	2.6652	3953	2.5300	4156	2.4063	4362	2.2925	4571	2.1876	26
35	3755	2.6628	3956	2.5279	4159	2.4043	4365	2.2907	4575	2.1859	25
36	3759	2.6605	3959	2.5257	4163	2.4023	4369	2.2889	4578	2.1842	24
37	3762	2.6581	3963	2.5236	4166	2.4004	4372	2.2871	4582	2.1825	23
38	3765	2.6558	3966	2.5214	4169	2.3984	4376	2.2853	4585	2.1808	22
39	3769	2.6534	3969	2.5193	4173	2.3964	4379	2.2835	4589	2.1792	21
40	3772	2.6511	3973	2.5172	4176	2.3945	4383	2.2817	4592	2.1775	20
41	3775	2.6488	3976	2.5150	4180	2.3925	4386	2.2799	4596	2.1758	19
42	3779	2.6464	3979	2.5129	4183	2.3906	4390	2.2781	4599	2.1742	18
43	3782	2.6441	3983	2.5108	4187	2.3886	4393	2.2763	4603	2.1725	17
44	3785	2.6418	3986	2.5086	4190	2.3867	4397	2.2745	4607	2.1708	16
45	3789	2.6395	3990	2.5065	4193	2.3847	4400	2.2727	4610	2.1692	15
46	3792	2.6371	3993	2.5044	4197	2.3828	4404	2.2709	4614	2.1675	14
47	3795	2.6348	3996	2.5023	4200	2.3808	4407	2.2691	4617	2.1659	13
48	3799	2.6325	4000	2.5002	4204	2.3789	4411	2.2673	4621	2.1642	12
49	3802	2.6302	4003	2.4981	4207	2.3770	4414	2.2655	4624	2.1625	11
50	3805	2.6279	4006	2.4960	4210	2.3750	4417	2.2637	4628	2.1609	10
51	3809	2.6256	4010	2.4939	4214	2.3731	4421	2.2620	4631	2.1592	9
52	3812	2.6233	4013	2.4918	4217	2.3712	4424	2.2602	4635	2.1576	8
53	3815	2.6210	4017	2.4897	4221	2.3693	4428	2.2584	4638	2.1560	7
54	3819	2.6187	4020	2.4876	4224	2.3673	4431	2.2566	4642	2.1543	6
55	3822	2.6165	4023	2.4855	4228	2.3654	4435	2.2549	4645	2.1527	5
56	3825	2.6142	4027	2.4834	4231	2.3635	4438	2.2531	4649	2.1510	4
57	3829	2.6119	4030	2.4813	4234	2.3616	4442	2.2513	4652	2.1494	3
58	3832	2.6096	4033	2.4792	4238	2.3597	4445	2.2496	4656	2.1478	2
59	3835	2.6074	4037	2.4772	4241	2.3578	4449	2.2478	4660	2.1461	1
60	3839	2.6051	4040	2.4751	4245	2.3559	4452	2.2460	4663	2.1445	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	69°		68°		67°		66°		65°		′

′	25°		26°		27°		28°		29°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	4663	2.1445	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	60
1	4667	2.1429	4881	2.0488	5099	1.9612	5321	1.8794	5547	1.8028	59
2	4670	2.1413	4885	2.0473	5103	1.9598	5325	1.8781	5551	1.8016	58
3	4674	2.1396	4888	2.0458	5106	1.9584	5328	1.8768	5555	1.8003	57
4	4677	2.1380	4892	2.0443	5110	1.9570	5332	1.8755	5558	1.7991	56
5	4681	2.1364	4895	2.0428	5114	1.9556	5336	1.8741	5562	1.7979	55
6	4684	2.1348	4899	2.0413	5117	1.9542	5340	1.8728	5566	1.7966	54
7	4688	2.1332	4903	2.0398	5121	1.9528	5343	1.8715	5570	1.7954	53
8	4691	2.1315	4906	2.0383	5125	1.9514	5347	1.8702	5574	1.7942	52
9	4695	2.1299	4910	2.0368	5128	1.9500	5351	1.8689	5577	1.7930	51
10	4699	2.1283	4913	2.0353	5132	1.9486	5354	1.8676	5581	1.7917	50
11	4702	2.1267	4917	2.0338	5136	1.9472	5358	1.8663	5585	1.7905	49
12	4706	2.1251	4921	2.0323	5139	1.9458	5362	1.8650	5589	1.7893	48
13	4709	2.1235	4924	2.0308	5143	1.9444	5366	1.8637	5593	1.7881	47
14	4713	2.1219	4928	2.0293	5147	1.9430	5369	1.8624	5596	1.7868	46
15	4716	2.1203	4931	2.0278	5150	1.9416	5373	1.8611	5600	1.7856	45
16	4720	2.1187	4935	2.0263	5154	1.9402	5377	1.8598	5604	1.7844	44
17	4723	2.1171	4939	2.0248	5158	1.9388	5381	1.8585	5608	1.7832	43
18	4727	2.1155	4942	2.0233	5161	1.9375	5384	1.8572	5612	1.7820	42
19	4731	2.1139	4946	2.0219	5165	1.9361	5388	1.8559	5616	1.7808	41
20	4734	2.1123	4950	2.0204	5169	1.9347	5392	1.8546	5619	1.7796	40
21	4738	2.1107	4953	2.0189	5172	1.9333	5396	1.8533	5623	1.7783	39
22	4741	2.1092	4957	2.0174	5176	1.9319	5399	1.8520	5627	1.7771	38
23	4745	2.1076	4960	2.0160	5180	1.9306	5403	1.8507	5631	1.7759	37
24	4748	2.1060	4964	2.0145	5184	1.9292	5407	1.8495	5635	1.7747	36
25	4752	2.1044	4968	2.0130	5187	1.9278	5411	1.8482	5639	1.7735	35
26	4755	2.1028	4971	2.0115	5191	1.9265	5415	1.8469	5642	1.7723	34
27	4759	2.1013	4975	2.0101	5195	1.9251	5418	1.8456	5646	1.7711	33
28	4763	2.0997	4979	2.0086	5198	1.9237	5422	1.8443	5650	1.7699	32
29	4766	2.0981	4982	2.0072	5202	1.9223	5426	1.8430	5654	1.7687	31
30	4770	2.0965	4986	2.0057	5206	1.9210	5430	1.8418	5658	1.7675	30
31	4773	2.0950	4989	2.0042	5209	1.9196	5433	1.8405	5662	1.7663	29
32	4777	2.0934	4993	2.0028	5213	1.9183	5437	1.8392	5665	1.7651	28
33	4780	2.0918	4997	2.0013	5217	1.9169	5441	1.8379	5669	1.7639	27
34	4784	2.0903	5000	1.9999	5220	1.9155	5445	1.8367	5673	1.7627	26
35	4788	2.0887	5004	1.9984	5224	1.9142	5448	1.8354	5677	1.7615	25
36	4791	2.0872	5008	1.9970	5228	1.9128	5452	1.8341	5681	1.7603	24
37	4795	2.0856	5011	1.9955	5232	1.9115	5456	1.8329	5685	1.7591	23
38	4798	2.0840	5015	1.9941	5235	1.9101	5460	1.8316	5688	1.7579	22
39	4802	2.0825	5019	1.9926	5239	1.9088	5464	1.8303	5692	1.7567	21
40	4806	2.0809	5022	1.9912	5243	1.9074	5467	1.8291	5696	1.7556	20
41	4809	2.0794	5026	1.9897	5246	1.9061	5471	1.8278	5700	1.7544	19
42	4813	2.0778	5029	1.9883	5250	1.9047	5475	1.8265	5704	1.7532	18
43	4816	2.0763	5033	1.9868	5254	1.9034	5479	1.8253	5708	1.7520	17
44	4820	2.0748	5037	1.9854	5258	1.9020	5482	1.8240	5712	1.7508	16
45	4823	2.0732	5040	1.9840	5261	1.9007	5486	1.8228	5715	1.7496	15
46	4827	2.0717	5044	1.9825	5265	1.8993	5490	1.8215	5719	1.7485	14
47	4831	2.0701	5048	1.9811	5269	1.8980	5494	1.8202	5723	1.7473	13
48	4834	2.0686	5051	1.9797	5272	1.8967	5498	1.8190	5727	1.7461	12
49	4838	2.0671	5055	1.9782	5276	1.8953	5501	1.8177	5731	1.7449	11
50	4841	2.0655	5059	1.9768	5280	1.8940	5505	1.8165	5735	1.7437	10
51	4845	2.0640	5062	1.9754	5284	1.8927	5509	1.8152	5739	1.7426	9
52	4849	2.0625	5066	1.9740	5287	1.8913	5513	1.8140	5743	1.7414	8
53	4852	2.0609	5070	1.9725	5291	1.8900	5517	1.8127	5746	1.7402	7
54	4856	2.0594	5073	1.9711	5295	1.8887	5520	1.8115	5750	1.7391	6
55	4859	2.0579	5077	1.9697	5298	1.8873	5524	1.8103	5754	1.7379	5
56	4863	2.0564	5081	1.9683	5302	1.8860	5528	1.8090	5758	1.7367	4
57	4867	2.0549	5084	1.9669	5306	1.8847	5532	1.8078	5762	1.7355	3
58	4870	2.0533	5088	1.9654	5310	1.8834	5535	1.8065	5766	1.7344	2
59	4874	2.0518	5092	1.9640	5313	1.8820	5539	1.8053	5770	1.7332	1
60	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	5774	1.7321	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	5774	1.7321	6009	1.6643	6249	1.6003	6494	1.5399	6745	1.4826	60
1	5777	1.7309	6013	1.6632	6253	1.5993	6498	1.5389	6749	1.4816	59
2	5781	1.7297	6017	1.6621	6257	1.5983	6502	1.5379	6754	1.4807	58
3	5785	1.7286	6020	1.6610	6261	1.5972	6506	1.5369	6758	1.4798	57
4	5789	1.7274	6024	1.6599	6265	1.5962	6511	1.5359	6762	1.4788	56
5	5793	1.7262	6028	1.6588	6269	1.5952	6515	1.5350	6766	1.4779	55
6	5797	1.7251	6032	1.6577	6273	1.5941	6519	1.5340	6771	1.4770	54
7	5801	1.7239	6036	1.6566	6277	1.5931	6523	1.5330	6775	1.4761	53
8	5805	1.7228	6040	1.6555	6281	1.5921	6527	1.5320	6779	1.4751	52
9	5808	1.7216	6044	1.6545	6285	1.5911	6531	1.5311	6783	1.4742	51
10	5812	1.7205	6048	1.6534	6289	1.5900	6536	1.5301	6787	1.4733	50
11	5816	1.7193	6052	1.6523	6293	1.5890	6540	1.5291	6792	1.4724	49
12	5820	1.7182	6056	1.6512	6297	1.5880	6544	1.5282	6796	1.4715	48
13	5824	1.7170	6060	1.6501	6301	1.5869	6548	1.5272	6800	1.4705	47
14	5828	1.7159	6064	1.6490	6305	1.5859	6552	1.5262	6805	1.4696	46
15	5832	1.7147	6068	1.6479	6310	1.5849	6556	1.5253	6809	1.4687	45
16	5836	1.7136	6072	1.6469	6314	1.5839	6560	1.5243	6813	1.4678	44
17	5840	1.7124	6076	1.6458	6318	1.5829	6565	1.5233	6817	1.4669	43
18	5844	1.7113	6080	1.6447	6322	1.5818	6569	1.5224	6822	1.4659	42
19	5847	1.7102	6084	1.6436	6326	1.5808	6573	1.5214	6826	1.4650	41
20	5851	1.7090	6088	1.6426	6330	1.5798	6577	1.5204	6830	1.4641	40
21	5855	1.7079	6092	1.6415	6334	1.5788	6581	1.5195	6834	1.4632	39
22	5859	1.7067	6096	1.6404	6338	1.5778	6585	1.5185	6839	1.4623	38
23	5863	1.7056	6100	1.6393	6342	1.5768	6590	1.5175	6843	1.4614	37
24	5867	1.7045	6104	1.6383	6346	1.5757	6594	1.5166	6847	1.4605	36
25	5871	1.7033	6108	1.6372	6350	1.5747	6598	1.5156	6851	1.4596	35
26	5875	1.7022	6112	1.6361	6354	1.5737	6602	1.5147	6856	1.4586	34
27	5879	1.7011	6116	1.6351	6358	1.5727	6606	1.5137	6860	1.4577	33
28	5883	1.6999	6120	1.6340	6363	1.5717	6610	1.5127	6864	1.4568	32
29	5887	1.6988	6124	1.6329	6367	1.5707	6615	1.5118	6869	1.4559	31
30	5890	1.6977	6128	1.6319	6371	1.5697	6619	1.5108	6873	1.4550	30
31	5894	1.6965	6132	1.6308	6375	1.5687	6623	1.5099	6877	1.4541	29
32	5898	1.6954	6136	1.6297	6379	1.5677	6627	1.5089	6881	1.4532	28
33	5902	1.6943	6140	1.6287	6383	1.5667	6631	1.5080	6886	1.4523	27
34	5906	1.6932	6144	1.6276	6387	1.5657	6636	1.5070	6890	1.4514	26
35	5910	1.6920	6148	1.6265	6391	1.5647	6640	1.5061	6894	1.4505	25
36	5914	1.6909	6152	1.6255	6395	1.5637	6644	1.5051	6899	1.4496	24
37	5918	1.6898	6156	1.6244	6399	1.5627	6648	1.5042	6903	1.4487	23
38	5922	1.6887	6160	1.6234	6403	1.5617	6652	1.5032	6907	1.4478	22
39	5926	1.6875	6164	1.6223	6408	1.5607	6657	1.5023	6911	1.4469	21
40	5930	1.6864	6168	1.6212	6412	1.5597	6661	1.5013	6916	1.4460	20
41	5934	1.6853	6172	1.6202	6416	1.5587	6665	1.5004	6920	1.4451	19
42	5938	1.6842	6176	1.6191	6420	1.5577	6669	1.4994	6924	1.4442	18
43	5942	1.6831	6180	1.6181	6424	1.5567	6673	1.4985	6929	1.4433	17
44	5945	1.6820	6184	1.6170	6428	1.5557	6678	1.4975	6933	1.4424	16
45	5949	1.6808	6188	1.6160	6432	1.5547	6682	1.4966	6937	1.4415	15
46	5953	1.6797	6192	1.6149	6436	1.5537	6686	1.4957	6942	1.4406	14
47	5957	1.6786	6196	1.6139	6440	1.5527	6690	1.4947	6946	1.4397	13
48	5961	1.6775	6200	1.6128	6445	1.5517	6694	1.4938	6950	1.4388	12
49	5965	1.6764	6204	1.6118	6449	1.5507	6699	1.4928	6954	1.4379	11
50	5969	1.6753	6208	1.6107	6453	1.5497	6703	1.4919	6959	1.4370	10
51	5973	1.6742	6212	1.6097	6457	1.5487	6707	1.4910	6963	1.4361	9
52	5977	1.6731	6216	1.6087	6461	1.5477	6711	1.4900	6967	1.4352	8
53	5981	1.6720	6220	1.6076	6465	1.5468	6716	1.4891	6972	1.4344	7
54	5985	1.6709	6224	1.6066	6469	1.5458	6720	1.4882	6976	1.4335	6
55	5989	1.6698	6228	1.6055	6473	1.5448	6724	1.4872	6980	1.4326	5
56	5993	1.6687	6233	1.6045	6478	1.5438	6728	1.4863	6985	1.4317	4
57	5997	1.6676	6237	1.6034	6482	1.5428	6732	1.4854	6989	1.4308	3
58	6001	1.6665	6241	1.6024	6486	1.5418	6737	1.4844	6993	1.4299	2
59	6005	1.6654	6245	1.6014	6490	1.5408	6741	1.4835	6998	1.4290	1
60	6009	1.6643	6249	1.6003	6494	1.5399	6745	1.4826	7002	1.4281	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	59°		58°		57°		56°		55°		′

′	35°		36°		37°		38°		39°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	7002	1.4281	7265	1.3764	7536	1.3270	7813	1.2799	8098	1.2349	**60**
1	7006	1.4273	7270	1.3755	7540	1.3262	7818	1.2792	8103	1.2342	59
2	7011	1.4264	7274	1.3747	7545	1.3254	7822	1.2784	8107	1.2334	58
3	7015	1.4255	7279	1.3739	7549	1.3246	7827	1.2776	8112	1.2327	57
4	7019	1.4246	7283	1.3730	7554	1.3238	7832	1.2769	8117	1.2320	56
5	7024	1.4237	7288	1.3722	7558	1.3230	7836	1 2761	8122	1.2312	55
6	7028	1.4229	7292	1.3713	7563	1.3222	7841	1.2753	8127	1.2305	54
7	7032	1.4220	7297	1.3705	7568	1.3214	7846	1.2746	8132	1.2298	53
8	7037	1.4211	7301	1.3697	7572	1.3206	7850	1.2738	8136	1.2290	52
9	7041	1.4202	7306	1.3688	7577	1.3198	7855	1.2731	8141	1.2283	51
10	7046	1.4193	7310	1.3680	7581	1.3190	7860	1.2723	8146	1.2276	**50**
11	7050	1.4185	7314	1.3672	7586	1.3182	7865	1.2715	8151	1.2268	49
12	7054	1.4176	7319	1.3663	7590	1.3175	7869	1.2708	8156	1.2261	48
13	7059	1.4167	7323	1.3655	7595	1.3167	7874	1.2700	8161	1.2254	47
14	7063	1.4158	7328	1.3647	7600	1.3159	7879	1.2693	8165	1.2247	46
15	7067	1.4150	7332	1.3638	7604	1.3151	7883	1.2685	8170	1.2239	45
16	7072	1.4141	7337	1.3630	7609	1.3143	7888	1.2677	8175	1.2232	44
17	7076	1.4132	7341	1.3622	7613	1.3135	7893	1.2670	8180	1.2225	43
18	7080	1.4124	7346	1.3613	7618	1.3127	7898	1.2662	8185	1.2218	42
19	7085	1.4115	7350	1.3605	7623	1.3119	7902	1.2655	8190	1.2210	41
20	7089	1.4106	7355	1.3597	7627	1.3111	7907	1.2647	8195	1.2203	**40**
21	7094	1.4097	7359	1.3588	7632	1.3103	7912	1.2640	8199	1.2196	39
22	7098	1.4089	7364	1.3580	7636	1.3095	7916	1.2632	8204	1.2189	38
23	7102	1.4080	7368	1.3572	7641	1.3087	7921	1.2624	8209	1.2181	37
24	7107	1.4071	7373	1.3564	7646	1.3079	7926	1.2617	8214	1.2174	36
25	7111	1.4063	7377	1.3555	7650	1.3072	7931	1.2609	8219	1.2167	35
26	7115	1.4054	7382	1.3547	7655	1.3064	7935	1.2602	8224	1.2160	34
27	7120	1.4045	7386	1.3539	7659	1.3056	7940	1.2594	8229	1.2153	33
28	7124	1.4037	7391	1.3531	7664	1.3048	7945	1.2587	8234	1.2145	32
29	7129	1.4028	7395	1.3522	7669	1.3040	7950	1.2579	8238	1.2138	31
30	7133	1.4019	7400	1.3514	7673	1.3032	7954	1.2572	8243	1.2131	**30**
31	7137	1.4011	7404	1.3506	7678	1.3024	7959	1.2564	8248	1.2124	29
32	7142	1.4002	7409	1.3498	7683	1.3017	7964	1.2557	8253	1.2117	28
33	7146	1.3994	7413	1.3490	7687	1.3009	7969	1.2549	8258	1.2109	27
34	7151	1.3985	7418	1.3481	7692	1.3001	7973	1.2542	8263	1.2102	26
35	7155	1.3976	7422	1.3473	7696	1.2993	7978	1.2534	8268	1.2095	25
36	7159	1.3968	7427	1.3465	7701	1.2985	7983	1.2527	8273	1.2088	24
37	7164	1.3959	7431	1.3457	7706	1.2977	7988	1.2519	8278	1.2081	23
38	7168	1.3951	7436	1.3449	7710	1.2970	7992	1.2512	8283	1.2074	22
39	7173	1.3942	7440	1.3440	7715	1.2962	7997	1.2504	8287	1.2066	21
40	7177	1.3934	7445	1.3432	7720	1.2954	8002	1.2497	8292	1.2059	**20**
41	7181	1.3925	7449	1.3424	7724	1.2946	8007	1.2489	8297	1.2052	19
42	7186	1.3916	7454	1.3416	7729	1.2938	8012	1.2482	8302	1.2045	18
43	7190	1.3908	7458	1.3408	7734	1.2931	8016	1.2475	8307	1.2038	17
44	7195	1.3899	7463	1.3400	7738	1.2923	8021	1.2467	8312	1.2031	16
45	7199	1.3891	7467	1.3392	7743	1.2915	8026	1.2460	8317	1.2024	15
46	7203	1.3882	7472	1.3384	7747	1.2907	8031	1.2452	8322	1.2017	14
47	7208	1.3874	7476	1.3375	7752	1.2900	8035	1.2445	8327	1.2009	13
48	7212	1.3865	7481	1.3367	7757	1.2892	8040	1.2437	8332	1.2002	12
49	7217	1.3857	7485	1.3359	7761	1.2884	8045	1.2430	8337	1.1995	11
50	7221	1.3848	7490	1.3351	7766	1.2876	8050	1.2423	8342	1.1988	**10**
51	7226	1.3840	7495	1.3343	7771	1.2869	8055	1.2415	8346	1.1981	9
52	7230	1.3831	7499	1.3335	7775	1.2861	8059	1.2408	8351	1.1974	8
53	7234	1.3823	7504	1.3327	7780	1.2853	8064	1.2401	8356	1.1967	7
54	7239	1.3814	7508	1.3319	7785	1.2846	8069	1.2393	8361	1.1960	6
55	7243	1.3806	7513	1.3311	7789	1.2838	8074	1.2386	8366	1.1953	5
56	7248	1.3798	7517	1.3303	7794	1.2830	8079	1.2378	8371	1.1946	4
57	7252	1.3789	7522	1.3295	7799	1.2822	8083	1.2371	8376	1.1939	3
58	7257	1.3781	7526	1.3287	7803	1.2815	8088	1.2364	8381	1.1932	2
59	7261	1.3772	7531	1.3278	7808	1.2807	8093	1.2356	8386	1.1925	1
60	7265	1.3764	7536	1.3270	7813	1.2799	8098	1.2349	8391	1.1918	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	54°		53°		52°		51°		50°		′

′	40°		41°		42°		43°		44°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	8391	1.1918	8693	1.1504	9004	1.1106	9325	1.0724	9657	1.0355	**60**
1	8396	1.1910	8698	1.1497	9009	1.1100	9331	1.0717	9663	1.0349	59
2	8401	1.1903	8703	1.1490	9015	1.1093	9336	1.0711	9668	1.0343	58
3	8406	1.1896	8708	1.1483	9020	1.1087	9341	1.0705	9674	1.0337	57
4	8411	1.1889	8713	1.1477	9025	1.1080	9347	1.0699	9679	1.0331	56
5	8416	1.1882	8718	1.1470	9030	1.1074	9352	1.0692	9685	1.0325	55
6	8421	1.1875	8724	1.1463	9036	1.1067	9358	1.0686	9691	1.0319	54
7	8426	1.1868	8729	1.1456	9041	1.1061	9363	1.0680	9696	1.0313	53
8	8431	1.1861	8734	1.1450	9046	1.1054	9369	1.0674	9702	1.0307	52
9	8436	1.1854	8739	1.1443	9052	1.1048	9374	1.0668	9708	1.0301	51
10	8441	1.1847	8744	1.1436	9057	1.1041	9380	1.0661	9713	1.0295	**50**
11	8446	1.1840	8749	1.1430	9062	1.1035	9385	1.0655	9719	1.0289	49
12	8451	1.1833	8754	1.1423	9067	1.1028	9391	1.0649	9725	1.0283	48
13	8456	1.1826	8759	1.1416	9073	1.1022	9396	1.0643	9730	1.0277	47
14	8461	1.1819	8765	1.1410	9078	1.1016	9402	1.0637	9736	1.0271	46
15	8466	1.1812	8770	1.1403	9083	1.1009	9407	1.0630	9742	1.0265	45
16	8471	1.1806	8775	1.1396	9089	1.1003	9413	1.0624	9747	1.0259	44
17	8476	1.1799	8780	1.1389	9094	1.0996	9418	1.0618	9753	1.0253	43
18	8481	1.1792	8785	1.1383	9099	1.0990	9424	1.0612	9759	1.0247	42
19	8486	1.1785	8790	1.1376	9105	1.0983	9429	1.0606	9764	1.0241	41
20	8491	1.1778	8796	1.1369	9110	1.0977	9435	1.0599	9770	1.0235	**40**
21	8496	1.1771	8801	1.1363	9115	1.0971	9440	1.0593	9776	1.0230	39
22	8501	1.1764	8806	1.1356	9121	1.0964	9446	1.0587	9781	1.0224	38
23	8506	1.1757	8811	1.1349	9126	1.0958	9451	1.0581	9787	1.0218	37
24	8511	1.1750	8816	1.1343	9131	1.0951	9457	1.0575	9793	1.0212	36
25	8516	1.1743	8821	1.1336	9137	1.0945	9462	1.0569	9798	1.0206	35
26	8521	1.1736	8827	1.1329	9142	1.0939	9468	1.0562	9804	1.0200	34
27	8526	1.1729	8832	1.1323	9147	1.0932	9473	1.0556	9810	1.0194	33
28	8531	1.1722	8837	1.1316	9153	1.0926	9479	1.0550	9816	1.0188	32
29	8536	1.1715	8842	1.1310	9158	1.0919	9484	1.0544	9821	1.0182	31
30	8541	1.1708	8847	1.1303	9163	1.0913	9490	1.0538	9827	1.0176	**30**
31	8546	1.1702	8852	1.1296	9169	1.0907	9495	1.0532	9833	1.0170	29
32	8551	1.1695	8858	1.1290	9174	1.0900	9501	1.0526	9838	1.0164	28
33	8556	1.1688	8863	1.1283	9179	1.0894	9506	1.0519	9844	1.0158	27
34	8561	1.1681	8868	1.1276	9185	1.0888	9512	1.0513	9850	1.0152	26
35	8566	1.1674	8873	1.1270	9190	1.0881	9517	1.0507	9856	1.0147	25
36	8571	1.1667	8878	1.1263	9195	1.0875	9523	1.0501	9861	1.0141	24
37	8576	1.1660	8884	1.1257	9201	1.0869	9528	1.0495	9867	1.0135	23
38	8581	1.1653	8889	1.1250	9206	1.0862	9534	1.0489	9873	1.0129	22
39	8586	1.1647	8894	1.1243	9212	1.0856	9540	1.0483	9879	1.0123	21
40	8591	1.1640	8899	1.1237	9217	1.0850	9545	1.0477	9884	1.0117	**20**
41	8596	1.1633	8904	1.1230	9222	1.0843	9551	1.0470	9890	1.0111	19
42	8601	1.1626	8910	1.1224	9228	1.0837	9556	1.0464	9896	1.0105	18
43	8606	1.1619	8915	1.1217	9233	1.0831	9562	1.0458	9902	1.0099	17
44	8611	1.1612	8920	1.1211	9239	1.0824	9567	1.0452	9907	1.0094	16
45	8617	1.1606	8925	1.1204	9244	1.0818	9573	1.0446	9913	1.0088	15
46	8622	1.1599	8931	1.1197	9249	1.0812	9578	1.0440	9919	1.0082	14
47	8627	1.1592	8936	1.1191	9255	1.0805	9584	1.0434	9925	1.0076	13
48	8632	1.1585	8941	1.1184	9260	1.0799	9590	1.0428	9930	1.0070	12
49	8637	1.1578	8946	1.1178	9266	1.0793	9595	1.0422	9936	1.0064	11
50	8642	1.1571	8952	1.1171	9271	1.0786	9601	1.0416	9942	1.0058	**10**
51	8647	1.1565	8957	1.1165	9276	1.0780	9606	1.0410	9948	1.0052	9
52	8652	1.1558	8962	1.1158	9282	1.0774	9612	1.0404	9954	1.0047	8
53	8657	1.1551	8967	1.1152	9287	1.0768	9618	1.0398	9959	1.0041	7
54	8662	1.1544	8972	1.1145	9293	1.0761	9623	1.0392	9965	1.0035	6
55	8667	1.1538	8978	1.1139	9298	1.0755	9629	1.0385	9971	1.0029	5
56	8672	1.1531	8983	1.1132	9303	1.0749	9643	1.0379	9977	1.0023	4
57	8678	1.1524	8988	1.1126	9309	1.0742	9640	1.0373	9983	1.0017	3
58	8683	1.1517	8994	1.1119	9314	1.0736	9646	1.0367	9988	1.0012	2
59	8688	1.1510	8999	1.1113	9320	1.0730	9651	1.0361	9994	1.0006	1
60	8693	1.1504	9004	1.1106	9325	1.0724	9657	1.0355	1.000	1.0000	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	49°		48°		47°		46°		45°		′

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